Outline – Modeling II

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Hybrid Control Systems

• In practice, several analog controllers and a number of microcontrollers / control computers (which are synchronized by the same clock) may be utilized as a part of a complex control scheme.
• From the stand-point of control engineering, it is highly desirable to develop equivalent (discrete-time) transfer functions representing the essential features of such control systems.

Properties of Sampled Data Systems

Property 1: (sampled signal)

\[ Y(s) \rightarrow \frac{Y^*(s)}{T} \]

\[ Y(z) = Z\{Y^*(s)\} \]

Property 2: (re-sampled signal)

\[ Y(s) \rightarrow \frac{Y^*(s)}{T} \]

\[ Y^*(s) = [Y^*(s)]^* \]

\[ Y(z) = Z\{Y^*(s)\} \]

Properties (Cont’d)

Property 3: (summation of sampled signals)

\[ Y^*(s) = Y_1^*(s) \pm Y_2^*(s) \pm \cdots \pm Y_n^*(s) \]

\[ Y(z) = Z\{Y_1^*(s)\} \pm Z\{Y_2^*(s)\} \pm \cdots \pm Z\{Y_n^*(s)\} \]

\[ Y(z) = Y_1(z) \pm Y_2(z) \pm \cdots \pm Y_n(z) \]
Properties (Cont’d)

Property 4: (sampled response)

\[ Y(s) = G(s)X(s) \]
\[ Y'(s) = [G(s)X(s)]^* \]
\[ Y(z) = Z\{G(s)X(s)\} \]

Notice that \[ Y'(s) \neq G'(s)X'(s) \]

Property 5: (sampled impulse response)

\[ X(z) = Z\{X'(s)\} \]
\[ G(z) = Z\{G'(s)\} \]
\[ Y(z) = Z\{G(z)X(z)\} \]

Property 6: (cascaded impulse responses)

\[ Y'(s) = [G_n(s)...G_2(s)G_1(s)X'(s)]^* \]
\[ Y'(s) = [G_n(s)...G_2(s)G_1(s)]^*X'(s) \]
\[ Y(z) = Z\{G_n(s)...G_2(s)G_1(s)\}X(z) \]

Property 7: (sampled impulse response of a cascaded system)

\[ Y'(s) = [G_n(s)...G_2(s)G_1(s)X'(s)]^* \]
\[ Y'(s) = [G_n(s)...G_2(s)G_1(s)]^*X'(s) \]
\[ Y(z) = Z\{G_n(s)...G_2(s)G_1(s)\}X(z) \]

Example 1

- Consider the block diagram of the illustrated system. Obtain the following transfer functions:
  a) \( B(z)/R(z) \)
  b) \( Y(z)/R(z) \)

Solution – Part (a)

Using the Property 3:
\[ E'(s) = R'(s) - B'(s) \Rightarrow \]
\[ E(z) = R(z) - B(z) \] \( (1) \)

From Property 5:
\[ M'(s) = [G_c(s)E'(s)]^* = G_c'(s)E'(s) \Rightarrow \]
\[ M(z) = G_c(z)E(z) \] \( (2) \)

where \[ G_c(z) = Z\{G_c'(s)\} = Z\{G_c(s)\} \]

Similarly, the Property 7 yields
\[ B'(s) = [H(s)G_p(s)G_{zoh}(s)M'(s)]^* \]
\[ B'(s) = [H(s)G_p(s)G_{zoh}(s)]^*M'(s) \Rightarrow \]
\[ B(z) = G(z)M(z) \] \( (3) \)

where \[ G(z) = Z\{H(s)G_p(s)G_{zoh}(s)\} \]
Solution – Part (a) (Cont’d)

Combining Eqns. (1), (2), and (3) gives

\[ [1 + G(z)G_c(z)]B(z) = G(z)G_c(z)R(z) \]

Thus,

\[
\frac{B(z)}{R(z)} = \frac{G(z)G_c(z)}{1 + G(z)G_c(z)}
\]

where

\[ G_c(z) \doteq Z\{G_c(s)\}; \]
\[ G(z) \doteq Z\{H(s)G_p(s)G_{zoh}(s)\} \]

Solution – Part (b)

To obtain the desired transfer function \( Y(z)/R(z) \), a fictitious sampler must be added to the system.

\[ E^*(s) = R^*(s) - B^*(s) \Rightarrow E(z) = R(z) - B(z) \quad (4) \]

From Property 5:

\[ M^*(s) = [G_c(s)E^*(s)]^\dagger = G_c^*(s)E^*(s) \Rightarrow M(z) = G_c^*(z)E(z) \quad (5) \]

where \( G_c(z) \doteq Z\{G_c^*(s)\} \)

Similarly, the Property 7 yields

\[ Y^*(s) = [G_p(s)G_{zoh}(s)M^*(s)]^\dagger \]
\[ Y^*(s) = [G_p(s)G_{zoh}(s)]^\dagger M^*(s) \Rightarrow Y(z) = G_c(z)M(z) \quad (6) \]

where \( G_c(z) \doteq Z\{G_c^*(s)G_{zoh}(s)\} \)

Solution – Part (b) (Cont’d)

Finally, the Property 7 gives

\[ B^*(s) = [H(s)G_p(s)G_{zoh}(s)M^*(s)]^\dagger \]
\[ B^*(s) = [H(s)G_p(s)G_{zoh}(s)]^\dagger M^*(s) \Rightarrow B(z) = G(z)M(z) \quad (7) \]

where \( G(z) \doteq Z\{H(s)G_p(s)G_{zoh}(s)\} \)

Dividing Eqn. (7) by (6) leads to

\[ \frac{B(z)}{Y(z)} = \frac{G(z)}{G_c(z)} \doteq \frac{G_2(z)}{G_1(z)} \quad (8) \]
Combining Eqns. (4), (5), (6), and (8) gives
\[ Y(z) = G_1(z)G_e(z)[R(z) - G_2(z)Y(z)] \]
Thus,
\[ Y(z)[1 + G_1(z)G_2(z)G_e(z)] = G_1(z)G_e(z)R(z) \]
\[ \frac{Y(z)}{R(z)} = \frac{G_1(z)G_e(z)}{1 + G(z)G_e(z)} \]

where
\[ G_1(z) \doteq Z\{G_1(s)\}; \]
\[ G_2(z) \doteq Z\{G_2(s)G_{zh}(s)\}; \]
\[ G(z) \doteq Z\{H(s)G_p(s)G_{zh}(s)\}; \]

Obtain the transfer function \( Y(z)/R(z) \) for the system shown.

\[ E^*(s) = R^*(s) - B^*(s) \Rightarrow \]
\[ E(z) = R(z) - B(z) \quad (1) \]
\[ X^*(s) = [G_3(s)R^*(s)]^* = G_3^*(s)R^*(s) \Rightarrow \]
\[ X(z) = G_3(z)R(z) \quad (2) \]
\[ B^*(s) = [H(s)Y^*(s)]^* = H^*(s)Y^*(s) \Rightarrow \]
\[ B(z) = H(z)Y(z) \quad (3) \]
\[ Y^*(s) = [U(s) - V(s)]^* = U^*(s) - V^*(s) \]
\[ Y^*(s) = [G_1(s)G_2(s)E^*(s)]^* - [G_2(s)X^*(s)]^* \]
\[ Y^*(s) = [G_1(s)G_2(s)]^* E^*(s) - G_2^*(s)X^*(s) \Rightarrow \]
\[ Y(z) = G_{12}(z)E(z) - G_2(z)X(z) \quad (4) \]

where
\[ G_{12}(z) \doteq Z\{G_1(s)G_2(s)\}; \]

Combining Eqns. (1), (2), (3), and (4) yields
\[ Y(z) = G_{12}(z)[R(z) - H(z)Y(z)] - G_4(z)G_4(z)R(z) \]
Hence,
\[ Y(z)[1 + G_{12}(z)H(z)] = [G_{12}(z) - G_3(z)G_4(z)]R(z) \]
\[ \frac{Y(z)}{R(z)} = \frac{G_{12}(z) - G_3(z)G_4(z)}{1 + G_{12}(z)H(z)} \]

where
\[ G_{12}(z) \doteq Z\{G_1(s)G_2(s)\}; \]
Discretization Methods

• The design of control system consists of two main steps:
  – The mathematical model of the plant is obtained to analyze its behaviour.
  – Using the model, an appropriate controller is designed to get the desired response from the controlled system.
• In continuous-time domain, the dynamics of the plant are represented by a set of differential equations.

Discretization Methods (Cont’d)

• In control literature, there exist various approximation techniques to convert such equations conveniently into discrete-time forms without utilizing z-transforms.
• The methods discussed here approximate the time derivative \( \frac{d}{dt} \) in ODE using a corresponding difference equation:
  1. Forward difference (Euler’s rule)
  2. Backward difference
  3. Trapezoidal (integration) rule
    • Also known as Tustin or bilinear transformation

1) Forward Difference

Utilizing the definition of derivative, we have
\[
\frac{dx}{dt} = \frac{x((k+1)T) - x(kT)}{(k+1)T - kT} = \frac{x(k+1) - x(k)}{T}
\]
In terms of forward time-shift operator \( q \):
\[
\frac{dx}{dt} = \frac{q - 1}{T} \cdot x(k) \Rightarrow \frac{d}{dt} \approx \frac{q - 1}{T}
\]
Since \( \frac{d}{dt} \) corresponds to \( s \) variable while \( q \) operator is equivalent to \( z \) variable, one can write
\[
s \approx \frac{z - 1}{T}
\]
The \( s \)-variable in the transfer function is conveniently replaced by this expression leading to a new transfer function of \( z \):
\[
G' (z) \approx G(s = \frac{z - 1}{T})
\]

2) Backward Difference

Utilizing the definition of derivative, we get
\[
\frac{dx}{dt} = \frac{x(kT) - x((k-1)T)}{kT - (k-1)T} = \frac{x(k) - x(k-1)}{T}
\]
In terms of backward time-shift operator \( q^{-1} \):
\[
\frac{dx}{dt} = \frac{1 - q^{-1}}{T} \cdot x(k) \Rightarrow \frac{d}{dt} \approx \frac{1 - q^{-1}}{T}
\]
As \( \frac{d}{dt} \) corresponds to \( s \) variable while \( q^{-1} \) operator is equivalent to \( z^{-1} \) variable, we have
\[
s \approx \frac{z - 1}{Tz}
\]
The \( s \)-variable in the transfer function is to be replaced by this expression leading to a new transfer function of \( z \):
\[
G' (z) \approx G(s = \frac{z - 1}{Tz})
\]
3) Trapezoidal (Tustin) Rule

Let \( u(t) = \frac{dx}{dt} \). The trapezoidal integration rule leads to
\[
x((k+1)T) = x(kT) + \frac{T}{2} [u((k+1)T) + u(kT)]
\]
\[
x((k+1)T) - x(kT) = \frac{T}{2} [u((k+1)T) + u(kT)]
\]

In terms of \( q \):
\[
(q-1)x(k) = \frac{T}{2} (q + 1) u(k)
\]
\[
u(k) = \frac{dx}{dt} \approx \frac{2}{T} \cdot \frac{q-1}{q+1} x(k) \Rightarrow \frac{d}{dt} \approx \frac{2}{T} \cdot \frac{q-1}{q+1}
\]

As \( \frac{d}{dt} \) corresponds to \( s \) variable while \( q \) operator is equivalent to \( z \) variable, we have
\[
s \approx \frac{2}{T} \cdot \frac{z-1}{z+1}
\]

Therefore,
\[
G'(z) \approx G(s = \frac{2}{T} \cdot \frac{z-1}{z+1})
\]

Alternative Derivation for Tustin (Bilinear) Transformation

Since \( z \approx e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \)
Recall that \( e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... \)

Then
\[
z \approx 1 + \frac{1}{1!} \left( \frac{sT}{2} \right) + \frac{1}{2!} \left( \frac{sT}{2} \right)^2 + ...
\]

If \( T \) is small, the higher order terms of the Taylor Series can be neglected:
\[
z \approx \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}
\]
Solving for \( s \) leads to
\[
s \approx \frac{2}{T} \cdot \frac{z-1}{z+1}
\]

Notes on Discretization Methods

- The discretization techniques discussed here approximate the dynamics of a continuous plant better when the sampling intervals are kept short.
  - Forward- and backward difference methods are quite vulnerable to long sampling periods.
  - Despite its complexity, the Tustin transformation yields the best results in most cases.
- Since short sampling periods are essential for the successful application of these techniques, the dynamics of output interfaces (like ZOH) could be neglected under such circumstances.

Example

Consider the ODE representing the dynamics of a plant:
\[
\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y(t) = x(t)
\]

If \( T = 0.01 \) s, obtain the discrete-time models for this system using the following techniques:
- Forward difference
- Backward difference
- Tustin transformation
- Z-transform with ZOH
Solution

Let us first develop the transfer function of this plant in s-domain:

\[(s^2 + 3s + 2)Y(s) = X(s) \Rightarrow \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2}\]

a) Forward Difference:

\[G'(z) = \frac{1}{\left(\frac{z}{T} + 3\left(\frac{z^-1}{T}\right)\right) + 2} = \frac{1}{z^2 - 1.9700z + 0.9702}\]

b) Backward Difference:

\[G'(z) = \frac{1}{\left(\frac{z}{T} + 3\left(\frac{z^-1}{T}\right)\right) + 2} = \frac{1}{z^2 - 1.9705z + 0.9707}\]

c) Tustin Transformation:

\[G'(z) \approx \frac{1}{\frac{1}{T^2}(1 - \frac{z^-1}{1 + \frac{z^-1}{2}}) + 2} = \frac{1}{z^2 - 1.9700z + 0.9704}\]

Be careful when rounding numbers in these transfer functions. Since we are dealing with relatively small numbers, make sure to perform all calculations to the highest precision and then display at least 4 digits after decimal point!

d) ZOH:

\[G(s) = \left(\frac{1}{s^2 + 3s + 2}\right) \frac{1 - e^{-sT}}{s} = \left(\frac{1}{s + 1} - \frac{1}{s + 2}\right) \frac{1 - e^{-sT}}{s}\]

\[G'(z) = Z\left[\left(\frac{1}{s + 1} - \frac{1}{s + 2}\right) \frac{1 - e^{-sT}}{s}\right] = (1 - z^-1)Z\left[\left(\frac{1}{s + 1} - \frac{1}{s + 2}\right) \frac{1}{s}\right]\]

From Z-transform tables:

\[G'(z) = z\left(\frac{1 - e^{-T}}{1 - z^{-1}e^{-T}}\right) \frac{1}{z \frac{1}{2} z^{-1} \left(1 - z^{-2T}\right)} \frac{1}{z \left(1 - z^{-2T}\right)} \frac{1}{z \left(1 - z^{-T}\right)}\]

\[G'(z) = \frac{z^{-1}(1 - e^{-T})}{1 - z^{-1}e^{-T}} \frac{1}{2} z^{-1}(1 - e^{-2T}) \frac{1}{1 - z^{-1}e^{-2T}} \frac{1}{1 - z^{-1}} \frac{1}{1 - e^{-T}}\]

Hence,

\[G'(z) = \frac{B_1 z^{-1} + B_2 z^{-2}}{1 - z^{-1}(e^{-T} + e^{-2T}) + z^{-2}e^{-3T}}\]

where

\[B_1 = (1 - e^{-T}) - \frac{1}{2}(1 - e^{-2T})\]

\[B_2 = \frac{1}{2} e^{-T} (1 - e^{-2T}) - e^{-2T} (1 - e^{-T})\]

In terms of numerical values, we have

\[G'(z) = \frac{4.9503 \times 10^{-5}z + 4.9010 \times 10^{-5}}{z^2 - 1.9702z + 0.9704}\]
Why Bother to Use Z-transforms?

- If compared to the discretization techniques discussed here, the model development efforts in the previous chapter appear to be tedious!
- However, notice that the discrete-time models (with ZOH) developed via z-transform techniques are NOT approximations.
- Independent of sampling rate, the outputs of such models exactly match to those of their continuous-time counterparts at sampling instances.

Zero-Order Hold

- Zero-order-hold (ZOH) is the interface between the digital and analog world.
  - Consists of a latch and a D/A converter:
    - Latch unit holds the binary signals for one sampling period (T),
    - D/A unit converts the binary representation into a corresponding output voltage.
  - Very simple and easy to implement!
- If the sampling rate is too slow, the output of a ZOH looks like a sequence of boxcars.
  - Some information is lost during the process,
  - A delay of (T/2) is introduced.

First-Order Hold (FOH)

- First-order-hold (FOH) and other higher-order methods are developed to overcome the representation difficulties associated with ZOH.
- FOH (or so called ramp-invariant hold) does a simple linear interpolation between m(k) and m(k+1).
  - It generates a corresponding output voltage m(t) when kT ≤ t ≤ (k+1)T.

Transfer Function of FOH

The FOH performs interpolation between sampling instances:

\[ m(t) = m(k) + \frac{t - kT}{T} \left[ m(k+1) - m(k) \right] \]

where \( kT \leq t \leq (k+1)T \). Without further elaboration, the transfer function of a FOH can be simply given as

\[ G_{\text{FOH}}(s) = \frac{(1 - e^{-sT})^2}{Te^{-sT}} \cdot \frac{1}{s^2} = \frac{(1 - z^{-1})^2}{Tz^{-1}} \cdot \frac{1}{s^2} \]

Since it requires the future (one-step ahead) value of its input m(k), it is very difficult to implement it in practice.
Practical Implementation of FOH

- Industry-standard "casual" realization of a FOH interface employs "extrapolation" techniques.
- When $kT < t < (k+1)T$, the slope of the output waveform is $\frac{m(k) - m(k-1)}{T}$.
- The corresponding transfer function of this practical implementation differs from the one given previously.

System Modeling with MATLAB

- Matlab provides excellent toolbox functions to model / design control systems.
- The following Matlab functions are commonly used for this purpose:
  - `tf`: to create the transfer functions of both continuous-time and discrete-time systems.
  - `c2d`: to convert continuous-time models into discrete-time ones.
  - `impulse/step`: to simulate the response of a system to impulse/step inputs.

Matlab Functions

To illustrate the use of these functions, let us revisit our previous example:

Transfer function:
\[
\frac{1}{s^2 + 3s + 2}
\]

Matlab's Output:
Transfer function:
\[
\frac{1}{s^2 + 3s + 2}
\]

One can define a transfer function in z-domain as well:

Transfer function:
\[
\frac{0.1z}{z - 0.9}
\]

Sampling time: 0.01

Matlab's Output:
Transfer function:
\[
0.1z
\]
\[
\frac{z - 0.9}{z}
\]

Sampling time: 0.01
Matlab Functions (Cont’d)

To convert the continuous-time model into a discrete-time one:

\[ G_{pd} = \text{c2d}(G_{pc}, 0.01, 'zoh') \]

Matlab’s Output:
Transfer function:
\[ \frac{4.95 \times 10^{-5} z + 4.901 \times 10^{-5}}{z^2 - 1.97 z + 0.9704} \]
Sampling time: 0.01

Other Matlab functions that might interest you at this point:
poly, roots, residue

Example

- If \( T = 0.01 \) s, obtain the transfer function \( B(z)/M(z) \) for the given system using MATLAB.

Solution

\[
\begin{align*}
\% \text{ Sample Matlab code} \\
\% \text{ G1 = tf(10,
\% [1 12]);} \\
\% \text{ G2 = tf(1,
\% [1 0]);} \\
\% \text{ H = tf([1] [1 5]);} \\
\% \text{ G = G1*G2*H;} \\
\% \text{ Gd = c2d(G, 0.01, 'zoh')} \\
\end{align*}
\]

Output:
Transfer function:
\[ \frac{1.598 \times 10^{-6} z^2 + 6.126 \times 10^{-6} z + 1.468 \times 10^{-6}}{z^3 - 2.838 z^2 + 2.682 z - 0.8437} \]
Sampling time: 0.01