Outline – Modeling

- Design Approaches
  - Design by Emulation
  - Direct Digital System Design
- Constant Coefficient Difference Equations
- Z-transforms
- Examples
- Properties
- Discrete-time System Modeling
- Examples:
  - Cruise Control
  - Water Tank
  - Motor Position Control

Modeling Issues

- Discrete-time:
  - *Constant Coefficient Difference Equations* (CCDEs) are used to model the systems.
  - No information on discrete variables is available between sampling instances.
  - Magnitudes of digital signals are quantized.
- Continuous-time:
  - *Ordinary Differential Equations* (ODEs) are used.
  - Time and every physical quantity are continuous.
  - Magnitudes are usually not quantized.

Design Issues

- How do we design a digital controller?
  - How to devise a control algorithm (CCDE) yielding the desired control performance?
- There are two design approaches to address this issue:
  - Design by emulation
  - Design for discrete-time domain

1 - Design by Emulation

- Treat controller as a *pseudo* continuous-time system:
  - Use control theory to design a continuous-time controller:
    - State-space
    - Laplace (frequency or s) domain
- Once such a controller is designed, implement its counterpart in discrete-time domain:
  - This approach relies on mapping techniques:
    - Assumes high sampling rates (small $T$!)
    - Not practical for many applications.
2 - Design for Discrete-time Domain

- Represent the plant as a discrete-time process:
  - Use discrete-time controller design techniques:
    - State-space techniques
    - z-domain techniques
- Most practical and reliable approach!

Constant Coefficient Difference Equations

All relationships in discrete-time domain can be expressed in terms of finite difference equations:

\[ y(k) = - \sum_{i=1}^{N} a_i y(k - i) + \sum_{j=0}^{M} b_j x(k - j) \quad (N \geq M) \]

For convenience, let us define a unit delay operator \( q^{-1} \):

\[
\begin{align*}
    y(k-1) &= q^{-1} y(k) \\
    y(k-2) &= q^{-2} y(k) \\
    &\vdots \\
    y(k-N) &= q^{-N} y(k)
\end{align*}
\]

\[
\begin{align*}
    y(k+1) &= q y(k) \\
    y(k+2) &= q^2 y(k) \\
    &\vdots \\
    y(k+N) &= q^{N} y(k)
\end{align*}
\]

In digital control literature, this operator \( q^{-1} \) is also known as backward time-shift operator.

CCDE (Cont’d)

Hence, the CCDE becomes

\[ y(k) = - \sum_{i=1}^{N} a_i q^{-i} y(k) + \sum_{j=0}^{M} b_j q^{-j} x(k) \]

The relationship between input \( x(k) \) and the output \( y(k) \) boils down to the ratio of two polynomials \( A(q^{-1}) \) and \( B(q^{-1}) \):

\[
\begin{align*}
    y(k) &= \frac{b_0 + b_1 q^{-1} + \ldots + b_M q^{-M}}{1 + a_1 q^{-1} + \ldots + a_N q^{-N}} = \frac{B(q^{-1})}{A(q^{-1})}
\end{align*}
\]

where \( a_1, \ldots, a_N, b_0, b_1, \ldots, b_N \in \mathbb{R} \) (Real numbers).
Z-transform of Sampled Signals

A discrete-time signal can be visualized as a sequence of binary numbers. As a mathematical abstraction, this sequence can be represented by a sum of impulse (Dirac-delta) functions:

\[
f'(t) = f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \ldots = \sum_{k=0}^{\infty} f(kT)\delta(t-kT)
\]

where

\[
\delta(t) = \begin{cases} 
\infty, & t = 0 \\
0, & t \neq 0
\end{cases}
\]

while

\[
\int_{0^-}^{0^+} \delta(t)dt = 1
\]

Using the transport delay property of Laplace transforms, we have

\[
L\{\delta(t-T)\} = e^{-sT} \cdot 1, \quad L\{\delta(t-2T)\} = e^{-2sT} \cdot 1, \ldots
\]

Therefore,

\[
F'(s) = L\{f'(t)\} = f(0) + f(T)e^{-sT} + f(2T)e^{-2sT} + \ldots
\]

Define a new complex variable as \(Z \equiv e^{sT}\):

\[
F(Z) = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \ldots = \sum_{k=0}^{\infty} f(kT)z^{-k}
\]

Consequently,

\[
F(Z) = Z\{f'(t)\} = Z\{f(kT)\}
\]

where \(Z\{f'(t)\}\) is called the \textit{Z-transform} of sampled time function \(f'(t)\).

Transport Delay Property

Recall that

\[
L\{f(t)\} = F(s)
\]

while

\[
L\{f(t-T)\} = e^{-sT}F(s)
\]

Conventions in Z-transforms

Z-transform is only defined for sampled time functions as

\[
F(z) = Z\{f(kT)\} = \sum_{k=0}^{\infty} f(kT)z^{-k}
\]

where \(z \equiv e^{sT}\).

However, the following conventions are commonly used for convenience:

\[
F(s) \xrightarrow{\text{Z Transform}} F(z) \equiv F(s) \xrightarrow{\text{Z Transform}} Z\{f(kT)\} \equiv F(z)
\]

\[
F(z) \xrightarrow{\text{Z Transform}} F(s) \equiv f(t) \xrightarrow{\text{Z Transform}} Z\{f(kT)\} \equiv F(z)
\]
Z-transform - Examples

1. Unit Impulse: \( f^\ast(t) = 1 \cdot \delta(t) + 0 + 0 + \ldots \)
   \[
   Z\{f^\ast(t)\} = Y(z) = 1 \cdot z^{-0} + 0 \cdot z^{-1} + 0 \cdot z^{-2} + \ldots = 1
   \]

2. Unit Step: \( f^\ast(t) = 1 \cdot \delta(t) + 1 \cdot \delta(t - T) + 1 \cdot \delta(t - 2T) + \ldots \)
   \[
   Z\{f^\ast(t)\} = 1 \cdot z^{-0} + z^{-1} + z^{-2} + \ldots = \frac{1}{1-z^{-1}}
   \]
   Recall that \( \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \) \( (a < 1) \)
   \[
   F(z) = \frac{1}{1-z^{-1}} \quad (|z^{-1}| < 1)
   \]

Examples (Cont’d)

3. Unit Ramp:
   \[
   f^\ast(t) = 0 \cdot \delta(t) + T \cdot \delta(t - T) + 2T \cdot \delta(t - 2T) + \ldots
   \]
   \[
   Z\{f^\ast(t)\} = 0 \cdot z^{-0} + T \cdot z^{-1} + 2T \cdot z^{-2} + \ldots = \sum_{k=0}^{\infty} (kT)z^{-k}
   \]
   \[
   F(z) = \frac{Tz^{-1}}{(1-z^{-1})^2}
   \]

Examples (Cont’d)

4. Arbitrary Function:
   \[
   f^\ast(t) = 0 \cdot \delta(t) + 1 \cdot \delta(t - T) + 2 \cdot \delta(t - 2T) + 2 \cdot \delta(t - 3T) + 1 \cdot \delta(t - 4T) + 0 + \ldots
   \]
   \[
   Z\{f^\ast(t)\} = 1 \cdot z^{-1} + 2 \cdot z^{-2} + 2 \cdot z^{-3} + 1 \cdot z^{-4}
   \]
   \[
   F(z) = z^{-1} \left[ 1 + 2z^{-1}(1 + z^{-1}) + z^{-3} \right]
   \]

Z-transform Table

<table>
<thead>
<tr>
<th>( F(s) )</th>
<th>( f(kT) )</th>
<th>( F(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta(kT) )</td>
<td>1</td>
</tr>
<tr>
<td>1/s</td>
<td>( 1(kT) )</td>
<td>( 1/(1-z^{-1}) )</td>
</tr>
<tr>
<td>1/s^2</td>
<td>( kT )</td>
<td>( Tz^{-1}/(1-z^{-1})^2 )</td>
</tr>
<tr>
<td>1/s^3</td>
<td>( (kT)^2/2! )</td>
<td>( T^2z^{-1}(1+z^{-1})/[2(1-z^{-1})^2] )</td>
</tr>
<tr>
<td>1/(s+\alpha)</td>
<td>( e^{-\alpha T} )</td>
<td>( 1/(1-z^{-1}e^{-\alpha T}) )</td>
</tr>
<tr>
<td>1/(s+\alpha)^2</td>
<td>( (kT)e^{-\alpha T} )</td>
<td>( z^{-1}Te^{-\alpha T}/(1-z^{-1}e^{-2\alpha T}) )</td>
</tr>
<tr>
<td>( a/[s(s+\alpha)] )</td>
<td>( 1 - e^{-\alpha T} )</td>
<td>( z^{-1}(1-e^{-\alpha T})/(1-z^{-1})(1-z^{-1}e^{-2\alpha T}) )</td>
</tr>
</tbody>
</table>
### Z-transform Table (Cont’d)

<table>
<thead>
<tr>
<th>( F(s) )</th>
<th>( f(kT) )</th>
<th>( F(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{a}{s^2(s+a)} )</td>
<td>((a^{-1})(akT-1+e^{-akT}))</td>
<td>( z^{-l}(A + Bz^{-1})/(a(1 - z^{-1})^2(1 - e^{-aT}z^{-1})^2) )</td>
</tr>
<tr>
<td>( \frac{s}{(s+a)^2} )</td>
<td>((1-akT)e^{-akT})</td>
<td>( 1 - z^{-1}(1+e^{-akT})/(1 - e^{-aT}z^{-1})^2 )</td>
</tr>
<tr>
<td>( \frac{s}{s^2+a^2} )</td>
<td>( \sin(akT))</td>
<td>( z^{-1}z^{-1}s^2 )</td>
</tr>
<tr>
<td>( \frac{s(a^2+a^2)}{a^2+b^2} )</td>
<td>( \cos(akT))</td>
<td>( z^{-1}e^{-aTz^{-1}}/[(1 + e^{-aT}z^{-1})^2] )</td>
</tr>
<tr>
<td>( \frac{b}{(s+a)^2+b^2} )</td>
<td>( \sin(akT) )</td>
<td>( z^{-1}e^{-aTz^{-1}}/[(1 + e^{-aT}z^{-1})^2] )</td>
</tr>
</tbody>
</table>

### Miscellaneous Issues in Z-transforms

- One can break up an unknown (Laplace) function into (a summation of) familiar functions that are likely to be found in the given tables using the following methods:
  - Partial fraction expansion or
  - Residue method
- Similar techniques could be employed to obtain the inverse Z-transform of an arbitrary function of \( z \).

### Properties of Z-transform

1. **Linearity:** \( Z\{a \cdot f(kT)\} = a \cdot Z\{f(kT)\} \)
2. **Addition:** \( Z\{f_1(kT) \pm f_2(kT)\} = Z\{f_1(kT)\} \pm Z\{f_2(kT)\} \)
3. **Time Shift:** \( Z\{f(kT - nkT)\} = z^{-kn}Z\{f(kT)\} \)
4. **Convolution:** \( Z\{\sum_{i=-\infty}^{\infty} f_i(iT)f_2(kT - iT)\} = F_1(z)F_2(z) \)
5. **Scaling:** \( Z\{r^{-k}f(kT)\} = F(rz) \)

### Properties (Cont’d)

6. **Final Value Theorem:** (valid only for stable systems)
\[ f(\infty) = \lim_{k \to \infty} f(kT) = \lim_{z \to 1} \frac{(z-1)}{z} F(z) \]
where \( F(z) = Z\{f(kT)\} \)
7. **Initial Value Theorem:**
\[ f(0) = \lim_{z \to \infty} F(z) \]
Discrete-time System Modeling

Assumptions in Digital Systems Modeling

• A/D and D/A conversions are non-linear operations:
  – Such converters cause a major difficulty in developing mathematical models in discrete-time domain.
• For the sake of convenience, these elements embedded inside the I/O interfaces are ignored.
  – Implies the use of high-resolution converters in the actual control system.
• Thus, only latch (i.e. zero-order hold) and sampler are taken into consideration.

Sampled Impulse Response

\[ y(t) = g(t) = L^{-1}\{G(s)\} \]

\[ y^*(t) = g(0) \cdot \delta(t) + g(T) \cdot \delta(t-T) + g(2T) \cdot \delta(t-2T) + \ldots + g(kT) \cdot \delta(t-kT) + \ldots \]

Similarly, \[ Y^*(s) = L\{y^*(t)\} = g(0) + g(T) e^{-sT} + g(2T) e^{-2sT} + \ldots \]

Z-transform of the sampled impulse response \( y^*(t) \) becomes

\[ Y(z) = g(0) + g(T) z^{-1} + g(2T) z^{-2} + \ldots = \sum_{k=0}^{\infty} g(kT) z^{-k} = G(z) \]

where \( G(z) \) is called the discrete-time transfer function of the system.
**Sampled Response to an Impulse Sequence**

The **Superposition principle** allows us to determine the overall response as a summation of individual impulse responses. For instance, when \( x(0) \delta(t) \) is applied alone, the corresponding output becomes

\[
y_0^*(t) = g(0)x(0)\delta(t) + g(T)x(0)\delta(t-T) + \ldots
\]

\[
\therefore Y_0(z) = G(z) \cdot x(0)
\]

Likewise, the response to \( x(T)\delta(t-T) \) is

\[
Y_1(s) = G(s)\left[ x(T) \cdot e^{-sT} \cdot 1 \right]_{\text{delayed impulse}}
\]

**Overall Response**

\[
Y(z) = Y_0(z) + Y_1(z) + \ldots
\]

\[
= G(z)\left[ x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots \right]_{Z\{x(kT)\} = X(z)}
\]

\[
Y(z) = G(z) \cdot X(z)
\]

\[
\therefore G(z) = \frac{Y(z)}{X(z)}
\]

**Response to a Sequence (Cont’d)**

\[
y_1(t) = L^{-1}\{x(T)e^{-sT}G(s)\} = x(T)g(t-T)
\]

where \( g(t) = L^{-1}\{G(s)\} \). Therefore, the sampled time-function is

\[
y_1^*(t) = g(-T) \cdot x(T) \cdot \delta(t) + g(0)x(T)\delta(t-T) + g(T)x(T)\delta(t-2T) + \ldots
\]

\[
Y_1(z) = 0z^{-0} + g(0)x(T)z^{-1} + \ldots = x(T)\sum_{k=0}^{\infty} \frac{g(kT)z^{-k}z^{-1}}{G(z)}
\]

\[
\therefore Y_1(z) = G(z) \cdot x(T) \cdot z^{-1}
\]

Similarly, \( Y_2(z) = G(z) \cdot x(T) \cdot z^{-2} \) and so on...

**Transfer Function of ZOH / Latch**

\[
x(t) \rightarrow x(kT)
\]

\[
x(s) \rightarrow G_{ZOH}(s)
\]

\[
y(s) \rightarrow y(kT)
\]

\[
x(t) = x(kT) e^{-skT}(1-e^{-sT})
\]

\[
x(t) = x(kT) e^{-skT}
\]

\[
x(t) = x(kT) e^{-skT}
\]
Transfer Function of ZOH (Cont’d)

Input function is the delayed impulse with a magnitude of \( x(kT) \):

\[
X(s) = x(kT) \cdot e^{-skT}
\]

Similarly, the output function becomes

\[
Y(s) = \frac{x(kT)}{s} \cdot e^{-skT}(1 - e^{-skT})
\]

Since

\[
G_{zoh}(s) = \frac{Y(s)}{X(s)} = \frac{1 - e^{-skT}}{s}
\]

Overall System Model

Operators / Complex Variables

**Continuous-time:**

<table>
<thead>
<tr>
<th>Laplace Domain</th>
<th>Time Domain</th>
<th>Physical Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>D operator</td>
<td>d/dt (derivative)</td>
</tr>
<tr>
<td>( 1/s )</td>
<td>1/D</td>
<td>( \int dt ) (integration in time)</td>
</tr>
</tbody>
</table>

**Discrete-time:**

<table>
<thead>
<tr>
<th>( z )-Domain</th>
<th>Time Domain</th>
<th>Physical Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z^{-1} )</td>
<td>( q^{-1} )</td>
<td>unit delay (delay by T)</td>
</tr>
</tbody>
</table>
Example 1 – Cruise Control

Consider the sports car shown.  

a) Develop a (simplified) equation of motion.  
b) Derive the corresponding CCDE.  
c) Simulate its response for a constant throttle.

Part (a) – Differential Equation

Ignoring rolling friction along with the cross-wind drag gives

\[ M \frac{dv}{dt} = \sum f = f_e - f_w \]

Assume that

\[ f_w = C \cdot v \]
\[ f_e = K_e \alpha \]

where \( \alpha \) is throttle angle; \( K_e \) is engine constant.

Part (a) – Transfer Function

Hence

\[ M \frac{dv}{dt} + Cv = K_e \alpha \]

Laplace transform of this ordinary differential equation becomes

\[ (Ms + C)V(s) = K_e A(s) \]

where \( V(s) = L\{v(t)\} \) and \( A(s) = L\{\alpha(t)\} \). Therefore, the transfer function is

\[ \frac{V(s)}{A(s)} = \frac{K_e}{Ms + C} \]

Part (b) – Discrete-time Model

Use the Z-transform table (see Slide 17):

\[ V(z) = Z\left(1 - z^{-1}\right)^{\frac{K_e}{C} s + \frac{\alpha}{C}} \]
\[ A(z) = \frac{K_e}{C} \left(1 - e^{-\alpha T} z^{-1}\right) \]

\[ V(z) A(z) = \frac{K_e}{C} \left(1 - e^{-\alpha T} z^{-1}\right) \left(1 - e^{-T z^{-1}}\right) \]
Part (b) (Cont’d)

\[ V(z) = \frac{K_c}{C}(1-e^{-aT})z^{-1} = b_1z^{-1} \]

\[ A(z) = \frac{1-e^{-aT}z^{-1}}{1-a_1z^{-1}} \]

\[ a_1 = e^{-CT/M} \]

\[ b_1 = \frac{K_c}{C}(1-a_1) \]

The corresponding CCDE is obtained by substituting

\[ z^{-1} \leftarrow q^{-1}; \quad V(z) \leftarrow v(k); \quad A(z) \leftarrow \alpha(k) \]

\[ \Rightarrow v(k) = a_1v(k-1) + b_1\alpha(k-1) \]

Part (c) – The Response

Let \( a_1 = 0.99; b_1 = 1; \alpha(k) = 0.1 \).

<table>
<thead>
<tr>
<th>k</th>
<th>( v(k) )</th>
<th>( v(k-1) )</th>
<th>( \alpha(k) )</th>
<th>( \alpha(k-1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.199</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>0.297</td>
<td>0.199</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>0.394</td>
<td>0.297</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Example 2 – Water Tank

- Consider the water tank illustrated.
  a) Obtain the differential equation.
  b) Draw the block diagram of the overall control system.
  c) Determine the discrete-time representation.
  d) If \( m(k) = 1 \) [V] (unit step) is applied, calculate the final water height.

One can easily compute this response with Matlab:

```matlab
>> alpha = 0.1*ones(700,1);
>> A = [1 -0.99]; B = [0 1];
>> v = filter(B,A,alpha);
>> stem(v)
```
Part (a) – Differential Equation

The differential equation of this system can be simply written as

\[ A \frac{dh}{dt} = q_i(t) - q_o(t) \]

Using the aforementioned relationships yields

\[ A \frac{db}{dt} + \frac{c_2}{c_3} b = c_1 m(t) \]

Taking the Laplace transform of this equation gives

\[ \frac{B(s)}{M(s)} = \frac{K}{s + a} \]

where \( a = \frac{c_2}{A} \) and \( K = \frac{c_1 c_3}{A} \).

Part (b) – Block Diagram

Part (c) – Discrete-time Model

The difference equation can be obtained conveniently by

\[ z^{-1} \leftarrow q^{-1}; \quad M(z) \leftarrow m(k); \quad B(z) \leftarrow b(k) \]

Hence,

\[ \frac{b(k)}{m(k)} = \frac{q^{-1} \beta_i}{1 - \alpha_i q^{-1}} \Rightarrow b(k)(1 - \alpha_i q^{-1}) = m(k)q^{-1} \beta_i \]

\[ b(k) = \alpha_i \cdot b(k-1) + \beta_i \cdot m(k-1) \]

where \( \alpha_i = e^{-aT} \) and \( \beta_i = (K/a)(1 - e^{-aT}) \).

Part (c) – Difference Equation

The difference equation can be obtained conveniently by

\[ \frac{B(z)}{M(z)} = \frac{K}{a} \left( 1 - z^{-1} \right) \frac{z^{-1}(1-e^{-at})}{(1-z^{-1})(1-z^{-1}e^{-at})} \]

\[ \frac{B(z)}{M(z)} = \frac{K}{a} \frac{z^{-1}(1-e^{-at})}{(1-z^{-1}e^{-at})} \]
Part (d) – Final Value

The z-transform of a unit step input is

\[ M(z) = \frac{1}{1-z^{-1}} \]

Hence, the measurement \( B(z) \) becomes

\[ B(z) = \frac{K}{c_3} \cdot \frac{z^{-1}(1-e^{-\alpha T})}{(1-z^{-1}e^{-\alpha T})} \cdot \frac{1}{1-z^{-1}} \]

Since \( B(z) = c_2H(z) \),

\[ H(z) = \frac{B(z)}{c_3} = \frac{K}{c_3} \cdot \frac{z^{-1}(1-e^{-\alpha T})}{(1-z^{-1}e^{-\alpha T})} \cdot \frac{1}{1-z^{-1}} \]

Let us use final value theorem (see Slide 21):

\[ h(\infty) = \lim_{z \to 1} \frac{z-1}{z} H(z) \]

\[ h(\infty) = \lim_{z \to 1} (1-z^{-1}) \cdot \frac{c_1}{c_2} \cdot \frac{z^{-1}(1-e^{-\alpha T/A})}{(1-z^{-1}e^{-\alpha T/A})} \cdot \frac{1}{1-z^{-1}} \]

The final value of \( h \) becomes

\[ h(\infty) = \frac{c_1}{c_2} \cdot \frac{1-e^{-\alpha T/A}}{1-e^{-\alpha T/A}} = \frac{c_1}{c_2} \]

When a step input is applied to the servo valve, the water level rises (exponentially) to its final level \( h(\infty) \).

Example 3 - Motor Control

- Consider the system shown.
  a) Develop \( \Theta(z)/M(z) \)
  b) Find \( \Theta(z)/\Omega(z) \)

```
\[
\Theta(z) = Z \left[ \frac{1-z^{-1}}{s} + 1 \cdot \frac{1}{s^2} \right] = \frac{1}{s^3} (1-z^{-1}) \]
\[
\Theta(z) = T^2 \left( \frac{1}{2s} \right) (z^{-1} + z^{-2}) \frac{1}{(1-z^{-1})^2} \]
\[
\Theta(z) = T^2 \left( \frac{1}{2s} \right) \frac{1}{(1-z^{-1})^2} \]
```

Part (a) - Motor Control System

```
\[
\Theta(z) = Z \left[ \frac{1-z^{-1}}{s} + 1 \cdot \frac{1}{s^2} \right] = \frac{1}{s^3} (1-z^{-1}) \]
\[
\Theta(z) = T^2 \left( \frac{1}{2s} \right) (z^{-1} + z^{-2}) \frac{1}{(1-z^{-1})^2} \]
\[
\Theta(z) = T^2 \left( \frac{1}{2s} \right) \frac{1}{(1-z^{-1})^2} \]
```
Part (b)

To obtain $\Theta(z)/\Omega(z)$, let us determine the transfer function $\Omega(z)/M(z)$ first.

Even though the speed ($\Omega$) is NOT actually sampled in this configuration, a fictitious speed sensor is added so as to obtain the desired relationship.

\[
\frac{\Omega(z)}{M(z)} = Z\left\{ \frac{(1-z^{-1})}{s} \cdot \frac{1}{J} \cdot \frac{1}{1} \right\} = \frac{1}{J}(1-z^{-1})Z\left\{ \frac{1}{s^2} \right\}
\]

\[
\frac{\Theta(z)}{M(z)} = \frac{T}{J}(1-z^{-1}) \cdot z^{-1} (1-z^{-1})^2
\]

A Common Mistake

Students are oftentimes tempted to do the following:

Since $\Theta(s)/\Omega(s) = 1/s$,

\[
\frac{\Theta(z)}{\Omega(z)} = Z\left\{ \frac{1}{s} \right\} = \frac{1}{(1-z^{-1})^2}
\]