Chapter 3

Computer Implementation of 1D FEM

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These notes are prepared with the hope to be useful to those who want to learn and teach FEM. You are free to use them. Please send feedbacks to the above email address.
Summary of Chapter 2

- In Chapter 2 Ritz method was improved into FEM.
  - Solution is not global anymore.
  - Solution over each element is simple. Complicated 2D and 3D solutions on complex domains can be obtained, by using necessary number of elements.
  - Problems with multiple materials can be solved.
  - Approximation function selection is very well defined and independent of BCs.

- But the procedure of Chapter 2 still have difficulties.
  - Writing approximation functions one-by-one for each node is difficult. In 2D and 3D it’ll be even more difficult.
  - Approximation functions change when mesh changes.
  - Symbolic math is limited and costly.

Exercise: Modify Example2_1v2.m code so that it asks the user to enter $NE$ and works automatically. Using tic & toc commands, measure computation time for $NE = 5, 10, 50, 100, 1000$. Improve the speed of the code in any way you can.
What Is This Chapter About?

- We’ll **improve** the solution procedure of Chapter 2 so that
  - it is almost completely mesh independent
  - approximation function calculations are easy
  - symbolic calculations are avoided
  - solution is efficient and very algorithmic

- To achieve these we’ll use
  - **Elemental weak form**
  - **Master element concept**
  - **Gauss Quadrature** numerical integration
Node Based Integral Calculation of Chapter 2

- In FEM $\phi$’s have local support.
- For linear elements, $\phi$’s are nonzero over at most two elements.

Integral of the $i^{th}$ eqn. contains $\phi_i$ in all its terms (see slide 2-12).
- But $\phi_i$ is nonzero only over elements $e-1$ and $e$.
- Therefore integral calculations simplify as follows

$$I_i = \int_{\Omega} f(\phi_i) \, dx = \int_{\Omega^{e-1}} f(\phi_i) \, dx + \int_{\Omega^e} f(\phi_i) \, dx$$

Over the whole problem domain
Over element $e-1$ only
Over element $e$ only
Node Based Integral Calculation of Chapter 2 (cont’d)

• For a 5 node mesh

All integrals are

\[
I_1 = \int_{\Omega^1} f(\phi_1) \, dx
\]
\[
I_2 = \int_{\Omega^1} f(\phi_2) \, dx + \int_{\Omega^2} f(\phi_2) \, dx
\]
\[
I_3 = \int_{\Omega^2} f(\phi_3) \, dx + \int_{\Omega^3} f(\phi_3) \, dx
\]
\[
I_4 = \int_{\Omega^3} f(\phi_4) \, dx + \int_{\Omega^4} f(\phi_4) \, dx
\]
\[
I_5 = \int_{\Omega^4} f(\phi_5) \, dx
\]

• This is node based thinking.

• We evaluate the integral of each equation, which are associated with one node and one \( \phi \).

• In FEM codes we prefer element based operations.
New Element Based Integral Calculation

• Instead of thinking about each equation individually, **concentrate on elements** and determine the contribution of each element to each equation.

\[
I_1 = \int_{\Omega^1} f(\phi_1) \, dx
\]

\[
I_2 = \int_{\Omega^1} f(\phi_2) \, dx + \int_{\Omega^2} f(\phi_2) \, dx
\]

\[
I_3 = \int_{\Omega^2} f(\phi_3) \, dx + \int_{\Omega^3} f(\phi_3) \, dx
\]

\[
I_4 = \int_{\Omega^3} f(\phi_4) \, dx + \int_{\Omega^4} f(\phi_4) \, dx
\]

\[
I_5 = \int_{\Omega^4} f(\phi_5) \, dx
\]
Elemental Thinking

• For the following model DE

\[-\frac{d}{dx}\left(a \frac{du}{dx}\right) + b \frac{du}{dx} + cu = f, \quad 0 < x < L\]

weak form is

\[\int_0^L \left(a \frac{du}{dx} \frac{dw}{dx} + bw \frac{du}{dx} + cwu\right) dx = \int_0^L wf \ dx + \left[w a \frac{du}{dx}\right]_Q - \left[w a \frac{du}{dx}\right]_0\]

• In Chapter 2 FE solution of this resulted in a $NN \times NN$ global system

\[[K]{u} = \{F\} + \{Q\}\]

• The new idea is to
  • write weak form over each element individually
  • obtain elemental systems
  • add them up to get the global system
Consider the following linear element

\[ \int_{x_i}^{x_{i+1}} \left( a \frac{du}{dx} \frac{dw}{dx} + bw \frac{du}{dx} + cwu \right) dx = \int_{x_1^e}^{x_2^e} \frac{w f}{dx} dx + \left[ \frac{w du}{dx} \right]_{x_1^e}^{x_2^e} - \left[ \frac{w du}{dx} \right]_{x_1^e}^{x_2^e} \]

which will result in the following \( NN \times NN \) elemental system

\[ [K^e] \{u\} = \{F^e\} + \{Q^e\} \]
Elemental System

\[ [K^e]\{u\} = \{F^e\} + \{Q^e\}\]

- Elemental systems are **sparse**.
- On a mesh of 4 linear elements
Small Elemental Systems of Size $NEN \times NEN$

- Each elemental system contributes to only 2 eqns of the global system.
- It is better to think of elemental systems as $NEN \times NEN$, instead of $NN \times NN$ where $NEN$ is the number of element’s nodes (=2 for linear elements)

$$[K^e]\{u^e\} = \{F^e\} + \{Q^e\}$$

- Elemental stiffness matrix $NEN \times NEN$
- Elemental unknown vector $NEN \times 1$
- Elemental force vector $NEN \times 1$
- Elemental boundary term vector $NEN \times 1$

- For example for $e=3$, small elemental system is

$$\begin{bmatrix} K_{11}^3 & K_{12}^3 \\ K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{bmatrix} u_1^3 \\ u_2^3 \end{bmatrix} = \begin{bmatrix} F_1^3 \\ F_2^3 \end{bmatrix} + \begin{bmatrix} Q_1^3 \\ Q_2^3 \end{bmatrix}$$

Assembly operation
From Approximation Functions to Shape Functions

Node based thinking

Element based thinking

Shape functions

\[ h^e = x_{i+1} - x_i = x_2^e - x_1^e \]
Shape Functions

• Similar to $\phi$’s, shape functions also have the Kronecker-delta property

$$S_i^e(x_j^e) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

• For a linear element shape functions are

$$S_1^e = \frac{x_2^e - x}{h^e}, \quad S_2^e = \frac{x - x_1^e}{h^e}$$

• FE solution over element $e$ is

$$u^e = \sum_{j=1}^{NEN} u_j^e S_j^e$$
Shape Functions (cont’d)

- **Difficulties** with these shape functions are
  - For each element they will be different functions of $x$.
  - Integration over an element will have limits of $x^e_1$ and $x^e_2$, which are not appropriate for Gauss Quadrature integration.

- The cure is to use the concept of **master element**.
Master Element

• 1D, linear master element is defined using the local coordinate $\xi$ (ksi).

\[ S_1(\xi) = \frac{1 - \xi}{2} \]
\[ S_2(\xi) = \frac{1 + \xi}{2} \]

Superscript “e” is NOT necessary here

• For all linear elements in a 1D mesh, there is only a single master element.
• Master element has a length of 2.
• End points are $\xi = -1$ and $\xi = 1$, which are suitable for Gauss Quadrature.
If a mesh has only linear elements, then we only need to define 2 shape functions. This is a great simplification, but it comes with a price.

In order to express everything in the integrals in terms of $\xi$, we need to obtain the relation between the global $x$ coordinate and the local $\xi$ coordinate.

This relation is linear as shown

$$x = A\xi + B$$

Using the fact that end points of the actual element coincide with those of the master element, we get

$$x = \frac{h^e}{2} \xi + \frac{x_1^e + x_2^e}{2}$$

This relation is different for each and every element.
In the integrals of the weak form we have the first derivative of \( u \).

\[
u^e = \sum_{j=1}^{NEN} u_j^e S_j \quad \rightarrow \quad \frac{d u^e}{dx} = \sum_{j=1}^{NEN} u_j^e \frac{dS_j}{dx}
\]

Master element shape functions are written in terms of \( \xi \).
Therefore \( x \) derivatives should be expressed in terms of \( \xi \) derivatives.

\[
\frac{dS_j}{dx} = \frac{dS_j}{d\xi} \frac{d\xi}{dx} = \frac{S_j}{h_e} \quad \text{(using the boxed equation of the previous slide.)}
\]

\[
\frac{dS_1}{d\xi} = -0.5 \quad , \quad \frac{dS_2}{d\xi} = 0.5
\]

\[
J^e = \frac{dx}{d\xi} = \frac{h_e}{2} \quad \text{is the Jacobian of element e. It is the ratio of actual element’s length to the length of the master element.}
\]
Example 3.1

Solve the following problem using a uniform mesh of 4 linear elements of length \( h^e = 0.25 \).

\[
-\frac{d^2 u}{dx^2} - u = -x^2, \quad 0 < x < 1
\]

\[
u(0) = 0, \quad u(1) = 0
\]

- Elemental weak form is

\[
\int_{x_1^e}^{x_2^e} \left( \frac{du}{dx} \frac{dw}{dx} - uw \right) dx = \int_{x_1^e}^{x_2^e} -wx^2 dx + \left[ w \frac{du}{dx} \right]_{x_2^e} - \left[ w \frac{du}{dx} \right]_{x_1^e}
\]

- To get 2x2 elemental system of eqns, substitute the following approximate solution into the elemental weak form

\[
u = \sum_{j=1}^{NEN} u_j^e S_j
\]
Example 3.1 (cont’d)

\[
\int_{\Omega^e} \left[ \frac{d}{dx} \left( \sum u_j^e S_j \right) \right] dw - w \sum u_j^e S_j \right] dx = \int_{\Omega^e} -wx^2 dx + \left[ w \frac{du}{dx} \right]_{x_2^e} - \left[ w \frac{du}{dx} \right]_{x_1^e}
\]

- Elemental system is 2x2 and we need 2 weight functions to get it.
- In GFEM \( w_1 = S_1 \), \( w_2 = S_2 \)

Eqn 1:

\[
\int_{\Omega^e} \left[ \frac{d}{dx} \left( \sum u_j^e S_j \right) \right] dS_1 - S_1 \sum u_j^e S_j \right] dx = \int_{\Omega^e} -S_1 x^2 dx + \left[ \frac{S_1}{0} \frac{du}{dx} \right]_{x_2^e} - \left[ \frac{S_1}{0} \frac{du}{dx} \right]_{x_1^e}
\]

Eqn 2:

\[
\int_{\Omega^e} \left[ \frac{d}{dx} \left( \sum u_j^e S_j \right) \right] dS_2 - S_2 \sum u_j^e S_j \right] dx = \int_{\Omega^e} -S_2 x^2 dx + \left[ \frac{S_2}{Q_2^e} \frac{du}{dx} \right]_{x_2^e} - \left[ \frac{S_1}{0} \frac{du}{dx} \right]_{x_1^e}
\]

- In general the \( i^{th} \) eqn of element \( e \) is obtained by using \( w = S_i \)

Eqn i:

\[
\int_{\Omega^e} \left[ \left( \sum u_j^e \frac{dS_j}{dx} \right) \right] dS_i - S_i \sum u_j^e S_j \right] dx = \int_{\Omega^e} -S_i x^2 dx + Q_i^e
\]
Example 3.1 (cont’d)

- Change the integration parameter from $x$ to $\xi$ (refer to slide 3-16)

\[
\int_{-1}^{1} \left[ \left( \sum u_j^e \frac{dS_j}{dx} \right) \frac{dS_i}{dx} - S_i \left( \sum u_j^e S_j \right) \right] dx = \int_{-1}^{1} -S_i x^2 dx + Q_i^e
\]

\[
dx = J^e d\xi
\]

\[
x = \frac{h^e}{2} \xi + \frac{x_1^e + x_2^e}{2}
\]

- Take the summation sign outside the integral and take the integrand into $u_j^e$ paranthesis.

\[
\sum \int_{-1}^{1} \left( \frac{dS_i}{d\xi} \frac{1}{J^e} \frac{dS_j}{d\xi} \frac{1}{J^e} - S_i S_j \right) J^e d\xi \ u_j^e = \int_{-1}^{1} S_i f(\xi) J^e d\xi + Q_i^e
\]
Example 3.1 (cont’d)

Eqn i : \[ \sum \int_{-1}^{1} \left( \frac{dS_i}{d\xi} J^e \frac{1}{J^e} \frac{dS_j}{d\xi} J^e - S_i S_j \right) \right] J^e d\xi \ u^e_j = \int_{-1}^{1} S_i f(\xi) J^e d\xi + Q^e_i \]

- Summation sign is over \( j = 1, 2 \).
- \( i \) index also goes from 1 to 2.
- \( i = 1 \) gives the first equation, \( i = 2 \) gives the second equation.
- 2x2 elemental system is \( [K^e] \{u\} = \{F^e\} + \{Q^e\} \)
- We don’t need to do any calculations for \( Q^e_i \) values (Details will come).
For each element $h^e = 0.25$.

- **Jacobian** for each element is $J^e = h^e/2 = 0.125$
- All elements are linear. **Shape functions** and their derivatives are
  \[
  S_1 = \frac{1 - \xi}{2}, \quad S_2 = \frac{1 + \xi}{2}
  \]
  \[
  \frac{dS_1}{d\xi} = -0.5, \quad \frac{dS_2}{d\xi} = 0.5
  \]

- We need everything to evaluate the entries of $K^e$ and $F^e$ one-by-one for each element.
Example 3.1 (cont’d)

\[ K_{ij}^e = \int_{-1}^{1} \left( \frac{dS_i}{d\xi} \frac{1}{J^e} \frac{dS_j}{d\xi} \frac{1}{J^e} - S_i S_j \right) J^e d\xi \]

• For e=1

\[ K_{11}^1 = \int_{-1}^{1} \left( \frac{dS_1}{d\xi} \frac{1}{J^e} \frac{dS_1}{d\xi} \frac{1}{J^e} - S_1 S_1 \right) J^e d\xi = \frac{47}{12} \]

\[ K_{12}^1 = \int_{-1}^{1} \left( \frac{dS_1}{d\xi} \frac{1}{J^e} \frac{dS_2}{d\xi} \frac{1}{J^e} - S_1 S_2 \right) J^e d\xi = -\frac{97}{24} \]

\[ K_{21}^1 = K_{12}^1 \quad ([K^e] \text{ is symmetric. Interchange } i \text{ & } j \text{ and see}) \]

\[ K_{22}^1 = \int_{-1}^{1} \left( \frac{dS_2}{d\xi} \frac{1}{J^e} \frac{dS_2}{d\xi} \frac{1}{J^e} - S_2 S_2 \right) J^e d\xi = \frac{47}{12} \]

\[ K^1 = \begin{bmatrix} 47/12 & -97/24 \\ -97/24 & 47/12 \end{bmatrix} \]
Example 3.1 (cont’d)

• No need to calculate \([K^2], [K^3] \) or \([K^4]\).
• They will all be equal to \([K^1]\). This is a special case for this problem. Can you see why?

• Let’s start \(\{F^e\} \) calculations.

\[
F_i^e = \int_{-1}^{1} S_i f(\xi) J^e d\xi
\]

\[
f(\xi) = -[x(\xi)]^2
\]

• For \(e=1\):

\[
f = -\left[\frac{h^e}{2} \xi + \frac{x_1^e + x_2^e}{2}\right]^2 = -\left(\frac{\xi + 1}{8}\right)^2
\]

\[
F_1^1 = \int_{-1}^{1} S_1 f(\xi) J^e d\xi = -\frac{1}{768}, \quad F_2^1 = \int_{-1}^{1} S_2 f(\xi) J^e d\xi = -\frac{3}{768}
\]
Example 3.1 (cont’d)

• For e=2: \( f = -\left(\frac{\xi+3}{8}\right)^2 \)

\[
F_1^2 = \int_{-1}^{1} S_1 f(\xi) J^e d\xi = -\frac{11}{768}, \quad F_2^2 = \int_{-1}^{1} S_2 f(\xi) J^e d\xi = -\frac{17}{768}
\]

• For e=3: \( f = ? \) (find yourself)

\[
F_1^3 = \int_{-1}^{1} S_1 f(\xi) J^e d\xi = -\frac{33}{768}, \quad F_2^3 = \int_{-1}^{1} S_2 f(\xi) J^e d\xi = -\frac{43}{768}
\]

• For e=4: \( f = ? \) (find yourself)

\[
F_1^4 = \int_{-1}^{1} S_1 f(\xi) J^e d\xi = -\frac{67}{768}, \quad F_2^4 = \int_{-1}^{1} S_2 f(\xi) J^e d\xi = -\frac{81}{768}
\]
Example 3.1 (cont’d)

• Four elemental systems are

For e=1:
\[
\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = -\frac{1}{768} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} Q_1^1 \\ Q_2^1 \end{bmatrix}
\]

For e=2:
\[
\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} = -\frac{1}{768} \begin{bmatrix} 11 \\ 17 \end{bmatrix} + \begin{bmatrix} Q_1^2 \\ Q_2^2 \end{bmatrix}
\]

For e=3:
\[
\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{bmatrix} u_1^3 \\ u_2^3 \end{bmatrix} = -\frac{1}{768} \begin{bmatrix} 33 \\ 43 \end{bmatrix} + \begin{bmatrix} Q_1^3 \\ Q_2^3 \end{bmatrix}
\]

For e=4:
\[
\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} \begin{bmatrix} u_1^4 \\ u_2^4 \end{bmatrix} = -\frac{1}{768} \begin{bmatrix} 67 \\ 81 \end{bmatrix} + \begin{bmatrix} Q_1^4 \\ Q_2^4 \end{bmatrix}
\]
Example 3.1 (cont’d)

- Assemble elemental systems into 5x5 global system (see slide 3-9).

\[
\begin{bmatrix}
K_{11}^1 & K_{12}^1 & 0 & 0 & 0 \\
K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 \\
0 & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 & 0 \\
0 & 0 & K_{22}^3 + K_{11}^4 & K_{12}^4 & 0 \\
0 & 0 & 0 & K_{22}^4 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
\end{bmatrix}
= \begin{bmatrix}
F_1^1 \\
F_2^1 + F_1^2 \\
F_2^2 + F_1^3 \\
F_2^3 + F_1^4 \\
F_2^4 \\
\end{bmatrix}
+ \begin{bmatrix}
Q_1^1 \\
Q_2^1 + Q_1^2 \\
Q_2^2 + Q_1^3 \\
Q_2^3 + Q_1^4 \\
Q_2^4 \\
\end{bmatrix}
\]
Example 3.1 (cont’d)

- Put the numbers in to get

\[
\begin{bmatrix}
  94 & -97 & -97 & -97 & -97 \\
-97 & 94 + 94 & -97 & 94 + 94 & -97 \\
-97 & 94 + 94 & 94 & 94 + 94 & 94 \\
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5 \\
\end{bmatrix}
= -\frac{1}{768}
\begin{bmatrix}
  1 & 3 + 11 & 17 + 33 & 43 + 67 & 81 \\
\end{bmatrix}
+ \begin{bmatrix}
  Q_1^1 \\
  Q_2^2 + Q_1^1 \\
  Q_2^3 + Q_1^3 \\
  Q_2^4 + Q_1^4 \\
\end{bmatrix}
\]

- Balance of secondary variables:

\[
Q_2^1 + Q_1^2 = \left(\frac{du}{dx}\right)_{x_2^1} + \left(-\frac{du}{dx}\right)_{x_2^2} = 0
\]
\[
Q_2^2 + Q_1^3 = \left(\frac{du}{dx}\right)_{x_2^2} + \left(-\frac{du}{dx}\right)_{x_2^3} = 0
\]
\[
Q_3^3 + Q_1^4 = \left(\frac{du}{dx}\right)_{x_2^3} + \left(-\frac{du}{dx}\right)_{x_2^4} = 0
\]
Example 3.1 (cont’d)

• Global system is

$$\frac{1}{24} \begin{bmatrix} 94 & -97 \\ -97 & 188 \\ -97 & 188 \\ -97 & 188 \\ -97 & 94 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = -\frac{1}{768} \begin{bmatrix} 1 \\ 14 \\ 50 \\ 110 \\ 81 \end{bmatrix} + \begin{bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ Q_5 \end{bmatrix}$$

• $u_1$ and $u_5$ are known.

• Reduce the system by dropping the 1st and 5th equations.

$$\frac{1}{24} \begin{bmatrix} 188 & -97 \\ -97 & 188 \\ -97 & 188 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = -\frac{1}{768} \begin{bmatrix} 14 \\ 50 \\ 110 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

• FE solution is

$$\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -0.0232 \\ -0.0405 \\ -0.0392 \end{bmatrix}$$
Apply EBCs without Reduction

• Reduction is not easy to implement in a computer code.
• A simpler technique is to keep the 1st and 5th eqns, but modify them as follows

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5
\end{bmatrix}
= 
\begin{bmatrix}
U_1 \\
0 \\
0 \\
0 \\
U_5
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Equate diagonal entries to 1, and non-diagonal entries to zero

These are the specified values of \(u_1\) and \(u_5\)

Equate unknown \(Q\)'s to zero.

• **Disadvantages** are
  • symmetry of \([K]\) is lost.
  • an unnecessarily large system is solved.
Apply EBCs without Reduction (cont’d)

- A third alternative for EBCs modifies 1\textsuperscript{st} and 5\textsuperscript{th} eqns as follows

\[
\begin{bmatrix}
L \times K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
K_{51} & K_{52} & K_{53} & K_{54} & L \times K_{55}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2 \\
\mathbf{u}_3 \\
\mathbf{u}_4 \\
\mathbf{u}_5
\end{bmatrix}
= \begin{bmatrix}
L \times K_{11} \times U_1 \\
F_2 \\
F_3 \\
F_4 \\
L \times K_{55} \times U_5
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where $L$ is large enough number.

- If $L$ is large enough the 1\textsuperscript{st} and 5\textsuperscript{th} eqns simplify to

\[
L K_{11} u_1 + \text{Negligibly small terms} = L K_{11} U_1 \quad \rightarrow \quad u_1 = U_1
\]

\[
L K_{55} u_5 + \text{Negligibly small terms} = L K_{55} U_5 \quad \rightarrow \quad u_5 = U_5
\]

- This technique preserves possible symmetry of $[K]$. 

**NBCs**

- If a NBC is provided, the specified $Q$ value is used in the global system.
- Similar to the Ritz method, NBCs are satisfied not exactly, but approximately.
- **Be careful in determining the SV correctly.**
- If a heat conduction problem is formulated starting from

\[-\frac{d}{dx} \left( kA \frac{dT}{dx} \right) + \ldots = 0\]

then $Q_1 = - \left( kA \frac{dT}{dx} \right)_0$ and $Q_{NN} = \left( kA \frac{dT}{dx} \right)_L$

- If in the same problem $kA$ is constant and dropped from the DE

\[-\frac{d}{dx} \left( \frac{dT}{dx} \right) + \ldots = 0\]

then $Q_1 = - \left( \frac{dT}{dx} \right)_0$ and $Q_{NN} = \left( \frac{dT}{dx} \right)_L$
**MBCs**

- Put the given mixed BC into the form

\[ SV = \alpha PV + \beta \]

where \( \alpha \) and \( \beta \) are known values.

- Use \( \alpha PV + \beta \) in the proper place of the \( \{Q\} \) vector.
- Transfer \( \alpha PV \) to the \([K]\) matrix and leave \( \beta \) on the RHS of the global system.

- If a mixed BC is given at the 5\(^{th}\) (last) node of a 4 element mesh

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5
\end{bmatrix} = 
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5
\end{bmatrix} + 
\begin{bmatrix}
Q_1 \\
0 \\
0 \\
0 \\
\alpha u_5 + \beta
\end{bmatrix}
\]

Modify \( K_{55} \) as

\( K_{55} = \alpha \)
GQ Integration (cont’d)

- In FEM integrals similar to the following ones need to be evaluated

\[ K_{ij}^e = \int_{-1}^{1} \left( \frac{dS_i}{d\xi} \frac{1}{J^e} \frac{dS_j}{d\xi} \frac{1}{J^e} - S_i S_j \right) J^e d\xi \quad , \quad F_{i}^e = \int_{-1}^{1} S_i f(\xi) J^e d\xi \]

- The limits [-1,1] are suitable for GQ integration, which converts an integral into a summation

\[ I = \int_{-1}^{1} g(\xi) d\xi = \sum_{k=1}^{NGP} g(\xi_k) W_k \]
Gauss Quadrature (GQ) Integration

- GQ points and weights for different $NGP$ values are

<table>
<thead>
<tr>
<th>$NGP$</th>
<th>$\xi_k$</th>
<th>$W_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>2.0</td>
</tr>
<tr>
<td>2</td>
<td>$-1/\sqrt{3} = -0.577350269189626$</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>$1/\sqrt{3} = 0.577350269189626$</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>$-\sqrt{0.6} = -0.774596669241483$</td>
<td>$5/9 = 0.5555555555555555$</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>$8/9 = 0.8888888888888889$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{0.6} = 0.774596669241483$</td>
<td>$5/9 = 0.5555555555555555$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.861136311594953$</td>
<td>0.347854845137454</td>
</tr>
<tr>
<td></td>
<td>$-0.339981043584856$</td>
<td>0.652145154862546</td>
</tr>
<tr>
<td></td>
<td>$0.339981043584856$</td>
<td>0.652145154862546</td>
</tr>
<tr>
<td></td>
<td>$0.861136311594953$</td>
<td>0.347854845137454</td>
</tr>
</tbody>
</table>

- $NGP$ point GQ integration can evaluate $(2NGP - 1)$ order polynomial functions exactly.
Example 3.2

Evaluate $K_{11}^1$ and $F_1^1$ of Example 3.1 using GQ integration.

$$K_{11}^1 = \int_{-1}^{1} \left( \frac{dS_1}{d\xi} \frac{1}{J^e} \frac{dS_1}{d\xi} \frac{1}{J^e} - S_1S_1 \right) J^e d\xi$$

$$= \int_{-1}^{1} \left[ (-0.5) \frac{1}{0.125} (-0.5) \frac{1}{0.125} - \left( \frac{1 - \xi}{2} \right) \left( \frac{1 - \xi}{2} \right) \right] (0.125) d\xi$$

$$= \int_{-1}^{1} \left( -\xi^2 + 2\xi + 63 \right) \frac{32}{g(\xi)} d\xi$$

- Using 1 point GQ: $K_{11}^1 = 2g(0) = 1.9688$
- Using 2 point GQ: $K_{11}^1 = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) = 3.9167$
- Using 3 point GQ: $K_{11}^1 = \frac{5}{9} g(-\sqrt{0.6}) + \frac{8}{9} g(0) + \frac{5}{9} g(\sqrt{0.6}) = 3.9167$

Both are exact
Example 3.2 (cont’d)

\[ F_1^1 = \int_{-1}^{1} S_1 f(\xi) J^e d\xi \]

\[ = \int_{-1}^{1} -\left(\frac{1 - \xi}{2}\right)\left(\frac{\xi + 1}{8}\right)^2 (0.125) \, d\xi \]

\[ = \int_{-1}^{1} \left(\frac{\xi^3 + \xi^2 - \xi - 1}{1024} \right) g(\xi) \, d\xi \]

- Using 1 point GQ: \( F_1^1 = 2g(0) = -0.0019531 \)
- Using 2 point GQ: \( F_1^1 = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) = -0.0013021 \)
- Using 3 point GQ: \( F_1^1 = \frac{5}{9} g\left(-\sqrt{0.6}\right) + \frac{8}{9} g(0) + \frac{5}{9} g\left(\sqrt{0.6}\right) = -0.0013021 \)

Both are exact