Nonlinear Control Systems
In real life most of the systems are nonlinear. Laplace transformation cannot be applied to nonlinear systems. Their analysis can be made only in the time domain. The nonlinearities may occur mainly in two places.

Plant Nonlinearities
Typical examples are:
- Dry (or Coulomb's) friction.
- Backlash
These can be illustrated as follows:

\[ f_b = bv \]
Viscous friction

\[ f_\mu = \mu N^* \text{sgn}(v) \]
dry friction
**Controller Non-linearities**

Typical examples
- Saturation which is unintentional
- Two-position control types which are intentional due to their simplicity

Saturation occurs if the control input is physically limited such that

\[ |u(t)| \leq u_m \quad \text{for all} \quad t \geq 0 \]

For example, P-control with saturation depends on the actuating error as follows

\[ u = K \varepsilon \quad \text{if} \quad |\varepsilon| < \varepsilon^* \]
\[ u = u_m \quad \text{if} \quad \varepsilon \geq \varepsilon^* \]
\[ u = -u_m \quad \text{if} \quad \varepsilon \leq -\varepsilon^* \]

\[ \varepsilon^* = u_m / K \]

Saturation is undesirable because the system may get uncontrollable or even unstable if the error exceeds a certain band, i.e. if \(|\varepsilon| \geq \varepsilon^*|\)

Note that \(\varepsilon^*\) can be increased by reducing \(K\), but this system sluggish in the proportional range. On the other hand to make the system responsive in the proportional range by increasing \(K\) reduces \(\varepsilon^*\) and this increases the possibility of getting saturated. So a compromising value for \(K\) should be selected.
As for the common intentional non-linear control types, the following ones are used in various applications due to their simplicities:

- **On-Off Control (without hysteresis)**

  
  ![Diagram of On-Off Control (without hysteresis)]

  
  \[ u = 0 \quad \text{if} \quad \varepsilon < 0 \]
  
  \[ u = u_m \quad \text{if} \quad \varepsilon > 0 \]
  
  \[ u = u_m \text{stp}(\varepsilon) \]
  
  \[ \text{stp}(\cdot) = \text{unit step function} \]

- **On-Off Control (with hysteresis)**

  
  ![Diagram of On-Off Control (with hysteresis)]

  
  \[ u = u_m \text{stp}(\varepsilon + \gamma) \quad \text{if} \quad \dot{\varepsilon} < 0 \]
  
  \[ u = u_m \text{stp}(\varepsilon - \gamma) \quad \text{if} \quad \dot{\varepsilon} > 0 \]

  
  \[ \text{Differential gap} = 2\gamma \]

- **Bang–bang Control**

  
  ![Diagram of Bang–bang Control]

  
  \[ u = u_m \text{sgn}(\varepsilon) \]
  
  \[ u = u_m \quad \text{if} \quad \varepsilon > 0 \]
  
  \[ u = -u_m \quad \text{if} \quad \varepsilon < 0 \]

  
  \[ \text{sgn}(\cdot) = \text{signum} \]
The most common two position control type is the on off control with hysteresis. This is because, the devices (e.g. a thermostat) which generate this control have a differential gap usually by their own nature. Another reason is that the differential gap has an effect increasing the life of the device although it also causes a larger error. The following example will illustrate these points.

Example:
Temperature control of a room by means of a thermostat.

\[ \begin{align*}
T(t) & \quad \text{T(t)} \\
\text{u} \quad \text{heater} \\
R & \quad \text{To} \\
q & \quad 0 \leq u \leq u_m
\end{align*} \]

The differential equation for \( T(t) \) is

\[ \tau \frac{dT}{dt} + T = T_0 + Ru \]
\[ \tau = RC \quad \text{Time constant} \]

\( T(t) \) is required to be kept close to a desired value \( T_r \) by regulating \( u \) with a thermostat. For simplicity let

\[ T' = T - T_0 \quad \text{and} \quad T'_r = T_r - T_0 \]

Then the differential equation will be

\[ \tau \frac{dT'}{dt} + T' = Ru \]

Cooling Phase

If the thermostat turns the heater off at \( t = t_i \), then for \( t_i \leq t \leq t_f \) before it turns on again; \( u = 0 \) and

\[ \tau \frac{dT'}{dt} + T' = 0 \quad \text{with} \quad T'(t_i) = T_i' \geq T'_r + \delta \]

The solution is

\[ T'(t) = T_i' e^{-\frac{t-t_i}{\tau}} \]
The thermostat will keep \( u = 0 \) until
\[ \varepsilon = T_r' - T' = \delta \quad \text{or} \quad T' = T'_r - \delta \]
Hence, \( t_j \) is found as follows
\[
T'_r - \delta = T'_i e^{-\left(\frac{\tau - t}{\tau}\right)} \quad t_j = t_i + \tau \ln \left( \frac{T'_r}{T'_r - \delta} \right)
\]
Note: \( t_j > t_i \) if \( T'_i > T'_r - \delta > 0 \)

Heating Phase

If the thermostat turns the heater on at \( t = t_j \), then for \( t_j \leq t < t_k \) before it turns it off again \( u = u_m \) and \( \tau \dot{T} + T' = R u_m \) with \( T'(t_j) = T'_j \leq T'_r - \delta \)
The solution is
\[
T'(t) = R u_m + \left( T'_j - R u_m \right) e^{-\left(\frac{t-t_j}{\tau}\right)}
\]

The thermostat will keep \( u = u_m \) until
\[ \zeta = T'_r - T' = -\delta \quad \text{or} \quad T' = T'_r + \delta \]
Hence, \( t_k \) is found as follows
\[ t_k = t_j + \tau \ln \left( \frac{T'_r - Ru_m}{T'_r + \delta - Ru_m} \right) \]

Note that \( t_k \) will be read, real, and greater than \( t_j \) if \( T'_j - Ru_m < T'_r + \delta - Ru_m < 0 \) or
If \( T'_j < T'_r + \delta \) and \( Ru_m > T'_r + \delta \)
If \( Ru_m < T'_r + \delta \), the heater will be indefinitely on. That is to get \( Ru_m > T'_r + \delta \)
we must have a powerful heater with large enough \( u_m \) and/or a good insulation with
enough \( R \)

**Sequential Operation**

The sequential combination of heating and cooling phases results in the following
variation \( T'(t) \):

As noticed; the steady state behavior of \( T' \) is oscillation about \( T'_r \) with amplitude
\( \gamma \). That is \( \epsilon = T'_r - T' \) changes between \( -\delta \) and \( +\delta \). So reducing \( \gamma \) reduces \( \epsilon \) but it also
increases the switching frequency of the thermostat and hence reduces its life.
Therefore \( \gamma \) should have a compromising value. In the limit if \( \delta \) is approximately
equal to zero, the thermostat becomes almost an on-off controller without hysteresis,
but in that case, although \( \epsilon \) is approximately equal to zero, the switching frequency
becomes almost infinite. This phenomena is called chattering and of course it is very
harmful for the life of the thermostat. The chattering case is illustrated below:
As for the thermostat itself, there are variety of designs based on the principle of expansion or elongation with temperature. Basic structure of a design using a bimetal strip is shown below:

![Diagram of a thermostat design using a bimetal strip]

The differential gap (25) arises because it takes a larger force to pluck the contact heads off than the force of attraction.