1) The open loop transfer function of the system can be determined as

$$G_p(s) = \frac{K_m}{Js^2 + bs + k}$$

b) The normalized form of the plant transfer function can be written as

$$G_p(s) = \frac{K_m}{\frac{1}{k} s^2 + \frac{b}{k} s + 1}$$

where the DC (open-loop) gain of the plant is

$$K_{OL} = \frac{K_m}{k}$$

Hence, by using the numerical values for the plant parameters, the open loop gain of the system is $K_{OL} = \frac{5}{10} = 0.05$.

Since the open loop transfer function of the plant does not include any free integrator, it is a type zero system.

c) If a proportional controller with $K_p = 100$ is used, then the open loop gain of the system becomes 5. Hence, the steady state error of the system to a unit step input becomes

$$e_{ss} = \frac{1}{1 + K_{OL}} = \frac{1}{1 + 5} = \frac{1}{6} = 0.167$$

d) In order to eliminate the steady state error to a step input, a proportional-plus-integral controller is utilized. Therefore, the open loop transfer function of the system is re-written as,

$$G_{OL}(s) = G_c(s)G_p(s) = \left(\frac{K_p}{K_i} s + 1\right) \left(\frac{K_m}{\frac{1}{k} s^2 + \frac{b}{k} s + 1}\right)$$

Hence, the DC gain of the compensated system becomes
As it is required to have a maximum of 10 percent steady state error to a ramp input, the integral gain of the controller is determined as follows,

\[ e_{ss} = \frac{1}{K_{OL}} \leq 0.1 \]

\[ \frac{1}{K_{OL}} = \frac{1}{K_i K_m} \leq 0.1 \]

\[ K_i \geq 10 \frac{k}{K_m} \rightarrow K_i \geq 10 \frac{10}{0.5} \]

\[ K_i \geq 200 \]

Hence, let \( K_i = 200 \).

It is also required to have a stability margin of \( \mu = 7.5 \). The closed-loop transfer function of the given unity feedback system can be written as follows,

\[ M(s) = \frac{G_{OL}(s)}{1 + G_{OL}(s)} \]

\[ M(s) = \frac{K_m s (K_p s + K_i)}{K_p s + K_i} \frac{1 + K_m s (K_p s + K_i)}{s (s^2 + bs + k)} \]

\[ M(s) = \frac{K_p s + K_i}{J s^3 + bs^2 + (k + K_m K_p) s + K_m K_i} \]

Where the characteristic polynomial of the closed loop transfer function is

\[ D(s) = J s^3 + bs^2 + (k + K_m K_p) s + K_m K_i \]

Then, replace \( s \) by \( z - \mu \) to yield to
With some algebraic manipulations,

\[ D(z) = J(z - \mu)^3 + b(z - \mu)^2 + (k + K_m K_p)(z - \mu) + K_m K_i \]

By using the plant parameters and letting \( \mu = 7.5 \), the above polynomial can be written as

\[ D(z) = Jz^3 + (-3J\mu + b)z^2 + (3J\mu^2 - 2b\mu + k + K_m K_p)z + (-J\mu^3 + b\mu^2 - (k + K_m K_p)\mu + K_m K_i) \]

To have a third order system be stable, the following condition should be satisfied,

\[ 0.0625(6.53 + 0.5K_p) \geq 0.015(41.17 - 3.75K_p) \]

\[ K_p \geq 2.39 \]

Hence, let \( K_p = 2.39 \).

Finally, the parameters of the proportional-plus-integral controller are determined to yield to

\[ G_c(s) = 2.39 + \frac{200}{s} \]
2)  

a) For a given first order system \( G(s) = K \frac{T_0 s + 1}{T s + 1} \)

- The final value of the system will be \( K \).
- The tangent drawn to the transient response at zero time will cross the final value at the time \( T \).
- The initial value of the system will be \( K \frac{T_0}{T} \).

For the systems given

<table>
<thead>
<tr>
<th></th>
<th>K</th>
<th>T</th>
<th>( K \frac{T_0}{T} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.5</td>
<td></td>
<td>( 0 \Rightarrow T_0 = 0 )</td>
</tr>
</tbody>
</table>

Then the transfer function are given as,

\[
G(s) = \frac{4}{0.5s + 1}
\]

![Step Response Graph](image)

For the second system given \( G(s) = K \frac{T_0 s + 1}{T s^2 + 2\zeta T + 1} \) for unit step response

- The final value of the system will be \( Kx_0 = K \).
- The initial slope of the system (for \( y_0 = 0 \)) will be \( \frac{T_0}{T} y_f \).

From the response it is seen that \( y_f = 2 \), thus \( K = 2 \)

and at time zero it is seen that the slope is zero, thus, \( T_0 = 0 \)

meaning that the system has no numerator dynamics.

Now that we are left with two unknowns \( \zeta \) and \( T \). From the graph it is seen that the response is oscillatory so the system is underdamped. For an underdamped
system with no numerator dynamics, the response is written as,

\[ y(t) = y_f \left( 1 - a_0 e^{-\zeta \omega_n t} \sin(\omega_d t + \psi) \right) \]

\[ a_0 = \frac{1}{\sqrt{1 - \zeta^2}} \]

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

\[ \psi = \tan^{-1} \left( \frac{\sqrt{1 - \zeta^2}}{\zeta} \right) \]

Note that the exponential profile that determines the oscillatory behavior can be written as

\[ y_{up}(t) = y_f \left( 1 + a_0 e^{-\zeta \omega_n t} \right) \text{ for the upper side} \]

\[ y_{lw}(t) = y_f \left( 1 - a_0 e^{-\zeta \omega_n t} \right) \text{ for the lower side} \]

It is seen that the initial value of the exponential profile at time zero, only depends on damping ratio \( \zeta \).

\[ y_{up}(t = 0) = y_f \left( 1 - a_0 e^{-\zeta \omega_n t} \right) = y_f(1 - a_0) \]

Thus \( a_0 \) so that damping ratio can be determined by looking at the initial value.

\[ y_{up}(t = 0) = 4.2 = 2 \left( 1 + \frac{1}{\sqrt{1 - \zeta^2}} \right) \rightarrow \zeta = 0.4 \]

The remaining system characteristic time constant can be found from the period of oscillations.

\[ \omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{2} = \pi \]

Then, the natural frequency and system characteristic time is found as

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \rightarrow \omega_n = \frac{\pi}{\sqrt{1 - 0.4^2}} = 3.43 \rightarrow T = \frac{1}{\omega_n} = 0.29 \]

Finally the system can be defined as

\[ G(s) = K \frac{T_0 s + 1}{T^2 s^2 + 2\zeta T s + 1} = \frac{2}{0.085s^2 + 0.232s + 1} \]

\[ 0.085s^2 + 0.232s + 1 \]
b) $G_1$

When the step response of the system is examined, it can be concluded that the system has numerator dynamics. Thus, the transfer function of this system will be in the form of $G_1(s) = K(T_0s + 1)/(Ts + 1)$.

The initial and final values and the initial slope can be determined by utilizing the step response of the system given below.

\[
\begin{align*}
y_0 &= 2 \\
y_f &= 1 \\
m_0 &= \frac{1.3 - 2}{60} = -0.0233
\end{align*}
\]

Determining the DC gain of the transfer function:

\[
K = \frac{y_f}{x_0} = 1/1 = 1
\]

Determining the denominator time constant:

\[
T = \frac{(y_f - y_0)}{m_0} = \frac{(1 - 2)}{(-0.0233)} = 42.86
\]

Determining the numerator time constant:

\[
T_0 = \frac{y_0T}{y_f} = \frac{2(42.86)}{1} = 85.72
\]

Then the transfer function becomes:

\[
G_1(s) = \frac{K(T_0s + 1)}{(Ts + 1)} = \frac{(85.72s + 1)}{(42.86s + 1)}
\]

\[\text{Step Response } G_1\]

- $G_2$

When the step response of the system is examined, it can be concluded that the system is a second order one and it has no numerator dynamics due to the fact that its initial slope is zero. Thus, the transfer function of this system will be in the form of

\[
G_2(s) = \frac{K\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}
\]

Since the final value of this second order underdamped system is 1, the DC gain
of the system is equal to 1. For determining the damping ratio, the envelope equation may be used. The upper envelope equation is

\[ y_u(t) = y_f(1 + a_0 e^{-\zeta \omega_n t}), \text{ where } a_0 = 1/\sqrt{1 - \zeta^2} \]

From the plot, \( a_0 = 1.005 \). Then the **damping ratio** is determined as

\[ \zeta = \sqrt{1 - \left(\frac{1}{a_0}\right)^2} \approx 0.1 \]

The **undamped natural frequency** of the system can be determined, by measuring the period of damped oscillations from the step response as shown in the below figure and determining the damped natural frequency.

\[ T_d = \frac{2\pi}{\omega_d} = 6 \text{ sec} \]
\[ \omega_d = 1.05 \text{ rad/sec} \]
\[ \omega_n = \omega_d/\sqrt{1 - \zeta^2} = 1.05 \text{ rad/sec} \]

Then the transfer function becomes:

\[ G_2(s) = \frac{K\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{1.1}{s^2 + 0.21s + 1.1} \]

\[ \text{c)} \]

• \( G_a \)

When the step response of the system is examined, it can be concluded that the system is a first order one with numerator dynamics due to initial jump. Hence, the transfer function of this system will be in the form

\[ G_a(s) = \frac{K(T_0 s + 1)}{(Ts + 1)} \]

Since the final value is 1, the **DC gain** of the system is \( K = 1 \).

The initial and final values and the initial slope can be determined by utilizing the step response of the system given below.

\[ y_0 = 1.5 \]
Determining the **DC gain** of the transfer function:

\[ K = \frac{y_f}{x_0} = \frac{0.5}{1} = 0.5 \]

Determining the **denominator time constant**:

\[ T = \frac{y_f - y_0}{m_0} = \frac{0.5 - 1.5}{-10} = 0.1 \]

Determining the **numerator time constant**:

\[ T_0 = \frac{y_0 T}{y_f} = \frac{1.5(0.1)}{0.5} = 0.3 \]

Then the transfer function becomes:

\[ G_1(s) = \frac{K(T_0 s + 1)}{(Ts + 1)} = \frac{0.5(0.3s + 1)}{(0.1s + 1)} \]

• **\( G_b \)**

When the step response of the system is examined, it can be concluded that the system is a second order undamped one without numerator dynamics. Hence, the transfer function of this system will be in the form

\[ G_b(s) = K \frac{\omega_n^2}{s^2 + \omega_n^2} \]

The period of oscillations is \( T = \frac{2\pi}{\omega_n} = 1.8 \). Hence, the undamped natural frequency of the system is \( \omega_n = 3.49 \). Next, since the amplitude of oscillations is about \( 0.33/2 = 0.165 \), the **DC gain of the system** is \( K = 0.165 \). Finally the transfer function of the system is determined as

\[ G_b(s) = \frac{2}{s^2 + 12} \]