**PROBLEM 1**

The characteristic polynomial of a system is as follows.

\[ D(s) = 2s^5 - 2s^4 - 2s + 2 \]

a) Comment on the stability of the system by Hurwitz test.

b) Determine the stability of the system and if exists the number of poles with positive real parts by using Routh stability criterion.

c) Determine all the poles of this system by using the Routh array. *No credits will be given to the answers directly supplied by using the calculators.*

**Solution:**

a) Since not all the coefficients are positive and some (for \( s^3 \) and \( s^2 \)) are missing, Hurwitz test states that the system is **not stable**.

b) Apply Routh array.

\[ D(s) = 2s^5 - 2s^4 - 2s + 2 = 2(s^5 - s^4 - s + 1), \text{ so apply Routh array to } D(s) = s^5 - s^4 - s + 1 \]

<table>
<thead>
<tr>
<th>( s^5 )</th>
<th>1</th>
<th>0</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^4 )</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>0</td>
<td>0</td>
<td>ROZ</td>
</tr>
<tr>
<td>( s^3')</td>
<td>-4</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( s^2 )</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( s^2')</td>
<td>( \varepsilon )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( s^1 )</td>
<td>4/( \varepsilon )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^0 )</td>
<td>1</td>
<td></td>
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</tbody>
</table>

The sign change from 1 to -1 in the first two elements of the first column indicates that the system is unstable. There are 2 sign changes in the first column which means that there are 2 poles with positive real parts.

c) Auxiliary polynomial: \( P(s) = -s^4 + 1 \) \( \Rightarrow \) \( \frac{dP(s)}{ds} = -4s^3 \)

Let \( t = s^2 \) \( \Rightarrow \) \( t^2 = 1 \) \( \Rightarrow \) \( t = \pm 1 \)

\( s^2 = 1 \) \( \Rightarrow \) \( s_{1,2} = \pm 1 \)

\( s^2 = -1 \) \( \Rightarrow \) \( s_{3,4} = \pm i \)

\[ D_t(s) = \frac{s^5 - s^4 - s + 1}{-s^4 + 1} = -(s - 1) \]

\( D_t(s) = 0 \) \( \Rightarrow \) \( s_5 = 1 \)

\( s_{1,2} = \pm 1, s_{3,4} = \pm i, s_5 = 1 \)
**PROBLEM 2**

A plant is going to be controlled by a P-controller of gain $K$ by using unity feedback. The block diagram representation of the closed-loop system is shown below. The system is subjected to a unit ramp input.

![Block Diagram](image)

a) Determine conditions on the proportional gain $K$ ($> 0$) to have a non-oscillatory closed-loop response with steady-state error less than 0.1.

b) Draw the locus of closed-loop poles in complex plane for $K \geq 0$ (root-locus) and determine the maximum possible stability margin and corresponding minimum proportional gain for the closed-loop system.

c) Design a feedforward controller to eliminate the finite steady-state error in ramp input.

**Solution:**

a) First determine the conditions on the controller parameters for stability. The characteristic polynomial is found as

$$D(s) = \text{Num}[1 + G_o(s)] = \text{Num} \left[ 1 + K \frac{1}{s(0.01s+1)} \right] = 0.01s^2 + s + K$$

Apply Hurwitz test (necessary and sufficient condition for stability for second order systems) to the denominator polynomial. Then, $K > 0$ for stability.

OLTF is $G_o(s) = \frac{K}{s(0.01s+1)}$. So, the system is TYPE 1. Type 1 systems subjected to ramp input have a finite steady-state error given by $e_{ss} = \frac{r_1}{K_v}$ where $r_1 = 1$ (since unit ramp) and $K_v = K_{OL} = K$.

Thus, $e_{ss} = 1/K$ and to have $e_{ss} < 0.1$, $K > 10$.

To have a non-oscillatory closed-loop response, $\zeta \geq 1$. Since $2\zeta \omega_n = 100$ and $\omega_n = \sqrt{100K}$, then $\frac{100}{2\sqrt{100K}} \geq 1$ and $K \leq 25$. Thus, $10 < K \leq 25$.

b) Since for small values of $K$, the two poles are real, and as $K$ increases they will be first multiple then complex conjugate with constant real part as shown in the following diagram, the maximum stability margin and minimum proportional gain occurs when the poles are multiple when $K = 25$.

The closed-loop poles of the system are:

$$p_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \times 0.01 \times K}}{2 \times 0.01} = -50 \pm \sqrt{2500 - 100K}$$

and for $K = 25$, $p_{1,2} = -50$.

Thus, $\mu_{\text{max}} = 50$

The stability margin is 50 for all values of $K \geq 25$.

![Root Locus](image)

c) The block diagram of the closed-loop system with feedforward controller is as follows.
\[ C(s) = L(s)R(s) + K[R(s) - C(s)] \] and \( E(s) = R(s)C(s) \). Thus,

\[ E(s) = \frac{s(0.01s+1) - L}{s(0.01s+1) + K} = \frac{s^2 + 100s - 100L}{s^2 + 100s + 100K} \]

So, \( G_o(s) = \frac{K + L}{s(0.01s+1) - L} \). Type number of the system should be at least 2 so that steady-state error due to a ramp input becomes zero. So, let \( L = s \), then \( G_o(s) = \frac{K + \frac{s}{0.01s^2}}{s} \) which is a type 2 system. Thus, \( L = s \)
PROBLEM 3

The unit-step response of a closed-loop control system with P-controller and unity-feedback is shown in the following figure.

![Response Graph]

a) Determine the followings with reasoning.
   i) Steady-state gain
   ii) Type number
   iii) Minimum possible order of the system
   iv) Existence of numerator dynamics
   v) Peak time
   vi) Percent overshoot

b) What is the closed-loop transfer function of this system? Assume minimum possible order.

It is known that the plant transfer function is in the following form.

\[ G_p(s) = \frac{a_2}{s^2 + a_1 s + a_2} \]

It is also known that the closed-loop transfer function is in the following form.

\[ K_p(s) = \frac{2}{s^2 + \zeta \omega_n s + \omega_n^2} \]

K_p = 0.9 since the steady-state gain is 0.9.

Solution:

a) Determine the followings with reasoning.
   i) Steady-state gain
      0.9 since the response goes to 0.9 for a unit step input.
   ii) Type number
      0 since there is finite steady-state error for the step input.
   iii) Minimum possible order of the system
      2 since the response is oscillatory.
   iv) Existence of numerator dynamics
      No since the initial slope of the response is zero.
   v) Peak time
      1.6 second as read from the response plot.
   vi) Percent overshoot
      37% since peak value is 1.23 and steady-state value is 0.9.

b) Assume the closed-loop transfer function in the following form.

\[ M(s) = \frac{K_p \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

K_p = 0.9 since the steady-state gain is 0.9.
Period of oscillations, $T_d=3.2\,\text{s}$ as read from the response plot. So, $\omega_d = \frac{2\pi}{T_d} = 1.96\,\text{rad/s}$

Alternatively, $t_p=1.6\,\text{s}$, so $\omega_d = \frac{\pi}{t_p} = 1.96\,\text{rad/s}$

Percent overshoot, $M_p=37\%$, so $\varepsilon_p = e^{-\frac{\pi}{\sqrt{1-\zeta^2}}} = 0.37$. Thus, $\zeta=0.3$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ which gives $\omega_n=2\,\text{rad/s}$. Thus, $M(s) = \frac{3.6}{s^2+1.2s+4}$

c) The block diagram representation of the closed-loop system is as follows.

\[
\begin{align*}
R(s) & \quad + \quad \sum \quad K_p \quad \left( \frac{s^2+a_2s+a_2}{s^2+1}\right) \quad C(s) \\
\end{align*}
\]

\[M(s) = \frac{K_p}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{a_2}{s^2+a_1s+a_2} = \frac{K_p a_2}{s^2+a_1s+(1+K_p)a_2}
\]

So, equating steady-state gains $K_o = \frac{K_p}{1+K_p}$ and $K_p = \frac{K_o}{1-K_o}$. Thus, $K_p = \frac{0.9}{1-0.9} \Rightarrow K_p=9$

d) $M(s) = \frac{3.6}{s^2+1.2s+4} = \frac{a_2}{s^2+a_1s+(1+K_p)a_2}$ \Rightarrow $a_1=1.2$, $a_2=0.4$

$t_p=1\,\text{s}$, so $\omega_d = \frac{\pi}{t_p} = 3.14\,\text{rad/s}$

\[M(s) = \frac{K_o\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{a_2}{s^2+a_1s+(1+K_p)a_2}
\]

Since $2\zeta\omega_n = a_1 = 1.2$ and $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{3.14}{\sqrt{1-0.6^2}}$. Thus, $\omega_n=3.2\,\text{rad/s}$.

Also, $\omega_n^2 = (1+K_p)a_2$, so $K_p = \frac{\omega_n^2}{a_2} - 1 = \frac{10.24}{0.4} - 1 \Rightarrow K_p = 24.6$