ME 304 CONTROL SYSTEMS  
Spring 2007  
Sections 03 and 04  

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SOLUTIONS TO MIDTERM EXAMINATION 2  

May 16, 2007  
Time Allowed: 90 minutes  
Open Notes and Books  
All questions are equally weighted

Student No. : ....................  
Name : .............................  
SURNAME : ..........................  
Signature : .......................... 

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PROBLEM 1

For the feedforward transfer functions,

i) \[ G_1(s) = \frac{K(s+2)(s-2)}{s^2 + 3} \]

ii) \[ G_2(s) = \frac{K(s+4)(s+5)(s-2)}{s^2 + 3} \]

determine the ranges of \( K (>0) \) for which the corresponding unity feedback system is

a) stable,

b) marginally stable,

c) unstable

SOLUTION:

i) The characteristic polynomial \( D_1(s) \) for \( G_1(s) \) is given as

\[ D_1(s) = s^2 + 3 + K(s+2)(s-2) = s^2 + 3 + K(s^2 - 4) = (1+K)s^2 + (3 - 4K) \]

If the Routh array is formed

<table>
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<tr>
<th>1+K</th>
<th>3–4K</th>
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<td>0</td>
<td>3–4K</td>
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a) There is no \( K \) value for which the system is stable since the \( s \) term is missing.
b) For \( 0 < K < 0.75 \) the system is marginally stable since signs of both \( 1+K \) and \( 3–4K \) are positive.
c) For \( K \geq 0.75 \) the system is unstable since \( 1+K > 0 \) and \( 3–4K \leq 0 \). When \( K = 0.75 \), the system has a double pole at the origin.

ii) The characteristic polynomial \( D_2(s) \) for \( G_2(s) \) is given as

\[ D_2(s) = s^2 + 3 + K(s+4)(s+5)(s-2) = Ks^3 + (7K+1)s^2 + 2Ks + (3 - 40K) \]

If the Routh array is formed as

<table>
<thead>
<tr>
<th>( K )</th>
<th>2K</th>
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<tr>
<td>( 7K+1 )</td>
<td>3–40K</td>
</tr>
<tr>
<td>( K(54K−1)/( 7K+1) )</td>
<td>( 3–40K )</td>
</tr>
</tbody>
</table>

a) With \( K > 0 \), for stability, Hurwitz test requires that

\( 7K+1 > 0 \) which is automatically satisfied since \( K > 0 \)

\( 3–40K > 0 \) \( \Rightarrow \) \( K < 3/40 = 0.075 \)

and the Routh criterion requires that

\( 7K+1 > 0 \) the same as Hurwitz test

\( 2K(7K+1) > K(3–40K) \) \( \Rightarrow \) \( 54K – 1 > 0 \) \( \Rightarrow \) \( K > 1/54 \)

\( 3–40K > 0 \) the same as Hurwitz test

Hence, for stability \( 0.0185 < K < 0.075 \).
b) For marginal stability, \( K = 0.0185 \) (one pole at the origin) or \( K = 0.075 \) (an imaginary conjugate pair of poles).
c) For instability, \( 0 < K < 0.0185 \) or \( K > 0.075 \)
PROBLEM 2
Consider the following feedback control system.

\[ R(s) + \frac{3}{s(s+1)} \rightarrow \frac{2}{s^2} \rightarrow C(s) \]

a) Determine the open loop transfer function as a ratio of two polynomials in s.
b) Determine the closed loop transfer function as a ratio of two polynomials in s.
c) Determine the type number of the system.
d) Determine the value of the steady-state error for an input \( r(t) = 5h(t) \).
e) Determine the value of the steady-state error for an input \( r(t) = 7h(t) \).

SOLUTION:
In order to determine OLTF and CLTF, the inner loop has to be eliminated first as
\[
\frac{3}{s(s+1)} \cdot \frac{3}{s(s+1)} = \frac{3s}{s^2(s+1) + 3} = \frac{3s}{s^3 + s^2 + 3}
\]
so that the block diagram of the system becomes

\[ R(s) + \frac{3s}{s^3 + s^2 + 3} \rightarrow \frac{2}{s^2} \rightarrow C(s) \]

a) Since it is a unity feedback system, the open loop transfer function \( G_{OL}(s) \) is the multiplication of feedforward transfer functions:
\[
G_{OL}(s) = \frac{3s}{s^3 + s^2 + 3} \cdot \frac{2}{s^2} = \frac{6}{s^3 + s^2 + 3}
\]
b) The closed loop transfer function \( G_{CR}(s) \) is obtained as
\[
G_{CR}(s) = \frac{6}{1 + \frac{6}{s(s^3 + s^2 + 3)}} = \frac{6}{s^3 + s^2 + 3 + 6} = \frac{6}{s^4 + s^3 + 3s + 6}
\]
c) The type number of the system is 1 due to a free s term in the denominator of the OLTF.
d) Note that \( s^2 \) term is missing in the characteristic polynomial (that is the denominator of the CLTF), hence it fails the Hurwitz test. Therefore, the system cannot be stable. For a system, which is not stable, the steady state error is meaningless. If one tries to form the Routh array as

\[
\begin{array}{ccc}
1 & 0 & 6 \\
1 & 3 & \\
-3 &
\end{array}
\]

The 3rd entry in the first column becomes negative indicating a sign change hence an unstable system. Therefore, the value of the steady-state error for an input \( r(t) = 5h(t) \) is infinite.
e) Same as part (d).
PROBLEM 3
Consider the following control system using a proportional plus derivative controller as the feedback control strategy.

Note that the closed-loop transfer function \( M_R(s) \) between the output \( C(s) \) and the reference input \( R(s) \) is

\[
M_R(s) = \frac{(K_p + K_ds)(s + 1)}{4s^2 + (5 + K_d)s + (1 + K_p)} \Rightarrow K_d(s^2 + (K_p + K_d)s + K_p) \quad \frac{1}{4s+1}
\]

a) Determine the relationship between the controller parameters \( K_p(>0) \) and \( K_d(>0) \) for which the response of the system is oscillatory and sketch the corresponding region in \( K_p \) vs \( K_d \) parameter plane showing all its particulars.

b) Is it possible to set the stability margin of this feedback system as large as desired? If yes, explain how. If no, determine the maximum possible stability margin.

c) How close does the response of this feedback system (to a unit step reference input) follow the input at steady-state? Suggest a way improve the steady-state accuracy explaining its reasons.

d) Does the response of this feedback system to a unit step reference input have an initial jump? Explain why. If there is such a jump, determine its value.

SOLUTION:

a) For an oscillatory response, the damping ratio has to be less than one. That is

\[
\zeta = \frac{5 + K_d}{2\sqrt{4(1 + K_p)}} < 1 \implies K_p > \left( \frac{5 + K_d}{4} \right)^2 - 1
\]

The sketch of the corresponding region is given at the figure left.

b) Yes, it is possible to set the stability margin of this feedback system as large as desired. Since this is second order system, its stability margin is given by its decay rate, which is expressed as

\[
\sigma = \zeta \omega_n = \frac{1}{2} \cdot \frac{5 + K_d}{4} = 0.125(5 + K_d)
\]

Hence, it can be set to any desired value by adjusting the controller parameter \( K_d \).

c) The final value of the unit step response can be obtained by using the final value theorem as

\[
c_{ss} = \lim_{s \to 0} [sC(s)] = \lim_{s \to 0} \left( \frac{(K_p + K_d)s(s + 1)}{4s^2 + (5 + K_d)s + (1 + K_p)} \right) \frac{1}{s} = \frac{K_p}{1 + K_p} \text{ increase } K_p \text{ to have } c_{ss} \approx 1
\]

d) The initial value of the unit step response can be obtained by using initial value theorem as

\[
c(0^+) = \lim_{s \to \infty} [sC(s)] = \lim_{s \to \infty} \left( \frac{(K_p + K_d)s(s + 1)}{4s^2 + (5 + K_d)s + (1 + K_p)} \right) \frac{1}{s} = \frac{K_d}{4}
\]