**PROBLEM 1:**

A plant is going to be controlled by a P-controller by using unity feedback. The transfer function of the plant is given as:

\[ G_p(s) = \frac{2}{s^2 + 4s - 5} \]

a) Determine the stability of the uncontrolled plant.

b) Determine conditions on the proportional gain \( K > 0 \) to have a stable closed-loop system.

c) What is the maximum possible stability margin and corresponding minimum proportional gain for the closed-loop system?

**Solution:**

a) To determine the stability of the plant, apply Hurwitz test (necessary and sufficient condition for stability for second order systems) to the denominator polynomial of the plant transfer function.

\[ D_p(s) = s^2 + 4s - 5 \]

Although all the coefficients exist, due to the sign differences (namely -5), the uncontrolled plant is unstable.

b) Determine conditions on the proportional gain \( K > 0 \) to have a stable closed-loop system.

Characteristic polynomial of the closed-loop system is:

\[ D(s) = \text{Num}[1+G_o(s)] = \text{Num}\left[1+K \frac{2}{s^2 + 4s - 5}\right] = s^2 + 4s - 5 + 2K \]

\[ D(s) = s^2 + 4s - 5 + 2K \]

So for stability \(-5+2K>0\) \(\Rightarrow K>2.5\)

c) What is the maximum possible stability margin and corresponding minimum proportional gain for the closed-loop system?

The closed-loop poles of the system are:

\[ p_{1,2} = -4 \pm \sqrt{4^2 - 4(2K - 5)} \]
\[ = -2 \pm \sqrt{4 - (2K - 5)} \]

Since for small values of \( K \), the two poles are real, and as \( K \) increases they will be first multiple then complex conjugate with constant real part as shown in the following diagram, the maximum stability margin and minimum proportional gain occurs when the poles are multiple. Thus,

\[ \mu_{\text{max}} = 2 \quad \text{and} \quad 4 - 2K + 5 = 0 \Rightarrow K = 4.5 \]

The stability margin is 2 for all values of \( K > 4.5 \)
**PROBLEM 2:**

The block diagram shown above represents a position control system, which involves a proportional control action supported by a velocity feedback. For this system, the transfer function between the controlled position \( x(t) \) and the reference position \( r(t) \) turns out to be

\[
\frac{X(s)}{R(s)} = \frac{K}{ms^2 + KTs + K}.
\]

a) Show that the type number of this system is \( N = 1 \).

b) Obtain the steady state error \( e_{ss} \) for a ramp input \( r(t) = v_0t \) and show that it is proportional to \( T \).

c) Since \( e_{ss} \) is proportional to \( T \), would you let \( T = 0 \) in order to make \( e_{ss} = 0 \)?

If "yes", why? If "no", why not?

**Solution:**

a) \[
\frac{E(s)}{R(s)} = 1 - \frac{X(s)}{R(s)} = 1 - \frac{K}{ms^2 + KTs + K} = \frac{ms^2 + KTs}{ms^2 + KTs + K} = \frac{1}{1 + G_0(s)}.
\]

\[
1 + G_0(s) = \frac{ms^2 + KTs + K}{ms^2 + KTs} \rightarrow G_0(s) = \frac{K}{ms^2 + KTs};
\]

\[
G_0(s) = \frac{K}{s(ms + KT)} = \frac{(1/T)}{s(1 + \frac{m}{KT}s)} \rightarrow N = 1 \quad \text{and} \quad K_{OL} = 1/T.
\]

b) \[
R(s) = \frac{v_0}{s^2} \rightarrow sE(s) = \frac{(ms + KTs)v_0}{ms^2 + KTs + K}.
\]

\[
e_{ss} = \lim_{s \to 0} sE(s) \rightarrow e_{ss} = v_0T.
\]

c) It is not allowable to let \( T = 0 \); because with \( T = 0 \), the characteristic polynomial becomes

\[
D(s) = ms^2 + K,
\]

which fails the Hurwitz test. Therefore, with \( T = 0 \), the system loses its asymptotic stability.
**PROBLEM 3:**

The unit step response of a system is given as

\[ y_{stp}(t) = [5(e^{-2t} - 1) + 2 \sin 5t]h(t) \]

where \( h(t) \) is the unit step function applied at \( t=0 \).

a) Show that the unit impulse response of this system is

\[ y_{imp}(t) = 10(\cos 5t - e^{-2t})h(t) \]

b) Explain why there are no impulsive terms in \( y_{imp}(t) \).

c) Determine the transfer function \( G_{yx}(s) \) for this system as a ratio of two polynomials in terms of \( s \).

d) Determine the expression for response \( y(t) \) of this system valid for all times \( -\infty < t < +\infty \) with zero initial conditions to an input specified as

\[ x(t) = 3\delta(t) + 6h(t) \]

where \( \delta(t) \) is the unit impulse function applied at \( t=0 \).

**Solution:**

a) The unit impulse response of this system can be obtained as

\[ y_{imp}(t) = \frac{dy_{stp}(t)}{dt} = \frac{d}{dt} \left\{ [5(e^{-2t} - 1) + 2 \sin 5t]h(t) \right\} \]

\[ = \frac{d}{dt} \left\{ [5(e^{-2t} - 1) + 2 \sin 5t]h(t) + \left[ 5(e^{-2t} - 1) + 2 \sin 5t \right] \frac{dh(t)}{dt} \right\} \]

\[ = (-10e^{-2t} + 10 \cos 5t)h(t) = 10(\cos 5t - e^{-2t})h(t) \quad \text{Q.E.D.} \]

b) There exists no impulsive terms in \( y_{imp}(t) \) since the coefficient of \( \delta(t) \) vanishes at \( t=0 \), due to the fact that the exists no jump in in \( y_{stp}(t) \) at \( t=0 \).

c) The transfer function of this system can be obtained as

\[ G(s) = \mathcal{L}\{[g(t)] = \mathcal{L}\{y_{imp}(t)\} = \mathcal{L}\{10(\cos 5t - e^{-2t})h(t)\} \]

\[ = 10\mathcal{L}\{(\cos 5t)h(t)\} - 10\mathcal{L}\{(e^{-2t})h(t)\} \]

\[ = \frac{10s}{s^2 + 25} - \frac{10}{s + 2} = \frac{10[s(s + 2) - (s^2 + 25)]}{(s + 2)(s^2 + 25)} \]

\[ \frac{10[2s - 25]}{(s + 2)(s^2 + 25)} = \frac{10(2s - 25)}{(s + 2)(s^2 + 25)} \]

d) In order to determine the response of the system to an input \( x(t) = 3\delta(t) + 6h(t) \), one needs to combine the impulse response and step response as

\[ y(t) = 3y_{imp}(t) + 6y_{stp}(t) = 3 \cdot 10(\cos 5t - e^{-2t})h(t) + 6 \cdot [5(e^{-2t} - 1) + 2 \sin 5t]h(t) \]

\[ = [12 \sin 5t + 30(\cos 5t - 1)]h(t) \]