ME 304 CONTROL SYSTEMS
Spring 2005
Sections 02, 03, 04

Course Instructors: Dr. Bülent E. Platin, Dr. Tuna Balkan, Dr. Kemal Ö zgören

FINAL EXAMINATION
SOLUTIONS

June 03, 2005
Time Allowed: 180 minutes
Open Notes and Books
All questions are equally weighted

List No. : .........................
Student No. : .....................
Name : ..............................
SURNAME : .........................
Signature : .........................

1
2
3
4
5
6
Σ
**PROBLEM 1:**

The figure at right shows the sketch of a polishing machine. The user pushes the machine forward with a force $F$ in order to drive it with a velocity $v$. Meanwhile, the motor of the machine generates a torque $T$ in order to rotate the polishing roller with an angular speed $\omega$.

The relevant parameters of the machine are given below:

- $m_1$: mass of the casing and the motor,
- $m_2$: mass of the roller,
- $m_{12} = m_1 + m_2$;
- $J_C$: moment of inertia of the roller about its mass center $C$,
- $R$: radius of the roller;

The interaction between the polishing roller and the polished surface is modeled as a *viscous friction* with coefficient $b$. However, the frictional forces between the surface and the front and rear supports of the machine are negligible.

a) Complete the following free body diagrams by adding the necessary moments and horizontal forces including the *frictional force* $F_b$. Don't show the vertical forces.

b) Write the right hand sides of the following elementary equations related to the system:

$m_1 \dot{v} = F - F'$,  
$m_2 \dot{v} = F' + F_b$,  
$J_C \dot{\omega} = T - RF_b$;  
$F_b = b(\omega - v)$.

c) Determine the constants in the following equation that relates $V(s)$ to $T(s)$ and $F(s)$.

$$V(s) = \frac{b_0}{s(a_1s + a_0)}T(s) + \frac{c_1s + c_0}{s(a_1s + a_0)}F(s).$$

We can determine the constants as described below:

$$m_1 s V(s) + m_2 s V(s) = F(s) + F_b(s) \quad \rightarrow \quad (m_1 s + b) V(s) = F(s) + bR \Omega(s).$$

$$J_C s \Omega(s) = T(s) - R b[R \Omega(s) - V(s)] \quad \rightarrow \quad (J_C s + bR^2) \Omega(s) = T(s) + R b V(s).$$

$$\Omega(s) = \frac{(m_1 s + b) V(s) - F(s)}{bR} = \frac{T(s) + R b V(s)}{J_C s + bR^2},$$

$$(J_C s + bR^2)(m_1 s + b) V(s) - (J_C s + bR^2) F(s) = bR T(s) + b^2 R^2 V(s),$$

$$(J_C m_{12} s^2 + bR^2 m_{12}s + J_C b s) V(s) = bR T(s) + (J_C s + bR^2) F(s),$$

$$V(s) = \frac{bR}{s[J_C m_{12}s + (J_C + m_{12} R^2)b]} T(s) + \frac{J_C s + bR^2}{s[J_C m_{12}s + (J_C + m_{12} R^2)b]} F(s).$$

Hence,

$$b_0 = bR; \quad c_1 = J_C, \quad c_0 = bR^2; \quad a_1 = J_C m_{12}, \quad a_0 = (J_C + m_{12} R^2) b.$$
PROBLEM 2:
A mechanical plant consisting of a pure inertia is subjected to a frequency response test and the Bode diagrams at right are obtained where the applied torque in N.m is the input and the angular velocity of the inertia in rad/s is the output.

a) Explain why the transfer function of the plant can be estimated as:
\[ G_p(s) = \frac{10}{s} \]
b) Identify the inertia in kg.m².

c) In a typical control application, it is desired to make the angular velocity of the inertia zero under the effect of a disturbance torque \( T_d \) acting directly on it. An incomplete block diagram representation of the proposed combined open-loop and closed-loop control system is shown below. Design an open-loop controller and complete the block diagram such that full disturbance rejection can be obtained. Select proportional closed-loop control strategy with unity angular velocity feedback. Assume that the disturbance torque is measured with a sensor of gain \( K_T \). \( T_{dm} \) is the measured disturbance torque and \( T \) is the total control torque.

Solution:

a) Slope of magnitude diagram is -20dB/decade. So an integrator is expected with a DC gain of \( K = M(1) = 10^{(20/20)} = 10 \) since the gain at 1 rad/s is 20 dB.

b) \( T(s) = J\Omega(s) \).
So, \( G_p(s) = \frac{\Omega(s)}{T(s)} = \frac{1}{Js} = \frac{10}{s} \)
Thus, \( \frac{1}{J} = 10 \) and \( J = 0.1 \) kg.m².

c) Since \( T_d \) is multiplied with \( K_T \) in the sensor, it should be divided by \( K_T \) and subtracted from the control signal because of positive \( T_d \) input to the plant. Thus, \( G_o(s) = -1/K_T \).
PROBLEM 3:

For the system shown at right, it can be shown that the transfer function between the output \( y(t) \) and the input \( x(t) \) can be expressed as

\[
G(s) = \frac{Y(s)}{X(s)} = \frac{A^2 R s}{A^2 R s + k}
\]

a) Determine the sensitivity expressions \( S_R^Y(s) \) and \( S_k^Y(s) \) between the system output \( Y(s) \) and system parameters \( R \) and \( k \), respectively.

b) Explain why this system approximates the differentiation operation on incoming mechanical signals \( x(t) \) for sinusoidal inputs with low frequencies such that the transfer function becomes

\[
G(s) = \frac{Y(s)}{X(s)} \approx \frac{A^2 R}{s}
\]

c) If the uncertainties on the valve resistance and spring stiffness values are \( \pm 0.8\% \) and \( \pm 0.5\% \), respectively, find the maximum possible uncertainty on the steady state value of the output \( y_{ss} \) for a unit ramp input.

Solution:

a) Using the definition of parameter sensitivity:

\[
S_R^Y(s) = \frac{R}{Y} \left( \frac{\partial Y}{\partial R} \right) = \frac{R}{G} \left( \frac{\partial G}{\partial R} \right) = \frac{A^2 R s}{A^2 s} \left[ \frac{A^2 s(A^2 R s + k) - A^2 R (A^2 s)}{(A^2 R s + k)^2} \right] = \frac{k}{A^2 R s + k}
\]

\[
S_k^Y(s) = \frac{k}{Y} \left( \frac{\partial Y}{\partial k} \right) = \frac{k}{G} \left( \frac{\partial G}{\partial k} \right) = \frac{k(A^2 R s + k)}{A^2 s} \left[ \frac{-A^2 R s}{(A^2 R s + k)^2} \right] = \frac{k}{A^2 R s + k}
\]

b) For low frequencies, the first term “\( A^2 R j \omega \)” of the denominator of sinusoidal transfer function \( G(j \omega) \) below can be neglected as compared to the second term “\( k \)” to give

\[
G(j \omega) = \frac{A^2 R j \omega}{A^2 R j \omega + k} \approx \frac{A^2 R j \omega}{k} \quad \Rightarrow \quad G(s) \approx \frac{A^2 R}{s}
\]

c) Using the final value theorem with \( X(s)=1/s^2 \) for a unit ramp input,

\[
y_{ss} = \lim_{s \to 0} [s Y(s)] = \lim_{s \to 0} [s G(s) X(s)] = \lim_{s \to 0} \left( \frac{A^2 R s}{A^2 R s + k} \left( \frac{1}{s^2} \right) \right) = \frac{A^2 R}{k}
\]

\[
\frac{\delta y_{ss}}{y_{ss}} = \left| S_R^Y(0) \left( \frac{\delta R}{R} \right) \right| + \left| S_k^Y(0) \left( \frac{\delta k}{k} \right) \right| = \left| (1)(\pm 0.008) \right| + \left| (-1)(\pm 0.005) \right| = 1.3\%
\]
**PROBLEM 4:**

A thermometer requires 2 minutes to indicate 98% of its steady state response to a step change in the temperature to its environment.

a) Assuming this thermometer as a first order system, estimate its time constant.

b) Give the transfer function of this thermometer with no numerator dynamics, if it indicates the correct constant environmental temperature at steady state.

c) Now consider a bath whose initial temperature is 20°C, but at t=0 its temperature starts rising linearly at a rate of 10°C/min as shown graphically at right. Suppose that this thermometer is located in an environment with a temperature 0°C initially. Then, it is suddenly immersed in this bath at t=0. How much error does the thermometer show at steady state?

**Solution:**

a) Since it takes approximately four times the time constant $T$ for the step response of a first order system to reach 98% of its steady state value, $T$ of this system should be

$$ T = \frac{2}{4} = 0.5 \text{ min} $$

b) Since the thermometer indicates the correct constant environmental temperature at steady state, its steady state gain is 1. With the time constant is 0.5 and no numerator dynamics, the transfer function of this thermometer should be

$$ G(s) = \frac{1}{0.5s + 1} $$

c) Note that the input is the temperature of the environment and it is specified as

$$ x(t) = (20 + 10t)h(t) $$

or in Laplace domain

$$ X(s) = \frac{20 + 10}{s} $$

Therefore, the output which is the temperature indicated by the thermometer can be obtained as

$$ Y(s) = G(s)X(s) = \left( \frac{1}{0.5s + 1} \right) \left( \frac{20 + 10}{s + \frac{10}{s^2}} \right) = \frac{20s + 10}{s^2(0.5s + 1)} $$

The error shown by the thermometer can be expressed as

$$ E(s) = Y(s) - X(s) = G(s)X(s) = \left( \frac{1}{0.5s + 1} \right) \left( \frac{20 + 10}{s + \frac{10}{s^2}} \right) - \left( \frac{20}{s + \frac{10}{s^2}} \right) $$

$$ = -0.5s \left( \frac{20 + 10}{s + \frac{10}{s^2}} \right) = -\frac{0.5s(20s + 10)}{s^2(0.5s + 1)} $$

Using the final value theorem, the value of the steady state error can be obtained as

$$ e_{ss} = \lim_{s \to 0} [sE(s)] = \lim_{s \to 0} \left\{ -\frac{0.5s(20s + 10)}{s^2(0.5s + 1)} \right\} = -5^\circ C $$
PROBLEM 5:

A closed-loop control system is subjected to a step response test and the following time response plot is obtained. It is assumed that the control system is second-order with no numerator dynamics.

a) Determine the maximum percent overshoot, peak time, steady-state gain, undamped natural frequency, and damping ratio from the plot.

b) The closed-loop transfer function of the system is expected in terms of controller parameters K and T as follows. Identify the settings of the controller parameters K and T.

\[ G(s) = \frac{K}{s^2 + (1 + KT)s + K} \]

c) It is desired to shape the time response by adjusting the controller parameters such that the peak time \( t_p = 1 \text{s} \) and the maximum percent overshoot \( M_p = 5\% \). However, it is found that, these two specifications can not be met simultaneously; and the controller parameter \( T \) can only be set to \( 1/K \). If the requirement on maximum percent overshoot is going to be satisfied only, determine the values of controller parameters K and T. What will be the corresponding value of peak the time \( t_p \)?

Solution:

a) Since the peak value of the response is 1.25 from the plot, \( M_p = \frac{1.25 - 1}{1} \times 100 = 25\% \)

Also, from the plot \( t_p = 1.7 \text{s} \). So \( \frac{\pi \zeta}{\sqrt{1-\zeta^2}} = -\ln(0.25) = 1.39 \Rightarrow \zeta = 0.4 \)

\[ t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 1.7 = \frac{\pi}{\omega_n \sqrt{1-0.4^2}} \Rightarrow \omega_n = 2\text{rad/s} \]

b) \( K = \omega_n^2 = 4 \) and \( 1 + KT = 2\zeta\omega_n \Rightarrow T = \frac{2\zeta\omega_n - 1}{K} = 0.15 \)

c) \( M_p = 0.05 \), so \( \frac{\pi \zeta}{\sqrt{1-\zeta^2}} = -\ln(0.05) = 3 \Rightarrow \zeta = 0.69 \) and \( T = 1/K \Rightarrow D(s) = s^2 + 2s + K \)

So, \( 2\zeta\omega_n = 2 \) and \( \zeta\omega_n = 1 \). Thus, \( \omega_n = 1.45\text{rad/s} \) and \( K = \omega_n^2 = 2.1 \), \( T = 1/K = 0.48 \)

\[ t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{1.45\sqrt{1-0.69^2}} = 3\text{s} \]
PROBLEM 6:

The block diagram shown above represents a velocity control system with a proportional controller. In addition to the reference input $v_r(t)$, the system is also under the effect of a disturbance input $f_d(t)$. Both of the inputs are sinusoidal as indicated below:

$$v_r(t) = V_{ro} \sin(\omega_1 t) \quad \text{and} \quad f_d(t) = F_{do} \sin(\omega_2 t).$$

a) Under the effect of both inputs, show that the steady state error can be expressed as

$$\epsilon(t) = M_1 V_{ro} \sin(\omega_1 t + \phi_1) - M_2 F_{do} \sin(\omega_2 t + \phi_2)$$

where

$$M_1 = M_1(\omega_1) = \frac{\omega_1}{\sqrt{\omega_1^2 + K^2}} \quad \text{and} \quad M_2 = M_2(\omega_2) = \frac{1}{\sqrt{\omega_2^2 + K^2}}.$$

b) What are the expressions for $\phi_1 = \phi_1(\omega_1)$ and $\phi_2 = \phi_2(\omega_2)$?

c) The reference input is specified as a generally low frequency signal with a frequency ranging as $0 \leq \omega_1 \leq 20$ rad/s. On the other hand, it is known that the disturbance input is a generally high frequency signal with a frequency ranging as $20$ rad/s $\leq \omega_2 \leq 40$ rad/s. As for the amplitudes of the inputs, they are indicated as $V_{ro} = 1$ m/s and $F_{do} = 1$ N. Based on this information, determine the range of the control gain $K$ so that the worst possible error is limited as $\epsilon_{\text{worst}} \leq 0.1$m/s.

Solution:

a) $V(s) = \frac{1}{s} \left\{ F_d(s) + K[V_r(s) - V(s)] \right\} \quad \rightarrow \quad V(s) = \frac{F_d(s) + KV_r(s)}{s + K}$;

$$E(s) = V_r(s) - V(s) \quad \rightarrow \quad E(s) = \frac{s}{s + K} V_r(s) - \frac{1}{s + K} F_d(s) = G_1(s)V_r(s) - G_2(s)F_d(s).$$

$$M_1 = \left| G_1(j\omega_1) \right| = \frac{\omega_1}{\sqrt{\omega_1^2 + K^2}}, \quad M_2 = \left| G_2(j\omega_2) \right| = \frac{1}{\sqrt{\omega_2^2 + K^2}}.$$

b) $\phi_1 = \angle G_1(j\omega_1) = \angle(j\omega_1) - \angle(j\omega_1 + K) = \frac{\pi}{2} - \tan^{-1}(\omega_1/K)$,

$\phi_2 = \angle G_2(j\omega_2) = \angle(1) - \angle(j\omega_2 + K) = -\tan^{-1}(\omega_2/K)$.

c) Note that $M_1$ increases as $\omega_1$ increases but $M_2$ decreases as $\omega_2$ increases. Moreover, the worst error combination occurs when $\sin(\omega_1 t + \phi_1) = \text{sgn}(V_{ro})$ and $\sin(\omega_2 t + \phi_2) = -\text{sgn}(F_{do})$. Therefore, the worst possible error is

$$\epsilon_{\text{worst}} = (M_1)_{\text{max}} |V_{ro}| + (M_2)_{\text{max}} |F_{do}| = M_1(20) |V_{ro}| + M_2(20) |F_{do}|.$$

$$\epsilon_{\text{worst}} = \frac{20}{\sqrt{400 + K^2}} + \frac{1}{\sqrt{400 + K^2}} \leq 0.1 \quad \rightarrow \quad 21 \leq 0.1\sqrt{400 + K^2};$$

$$210 \leq \sqrt{400 + K^2} \quad \rightarrow \quad (210)^2 \leq 400 + K^2 \quad \rightarrow \quad K \geq 209.05 \text{ Ns/m}.$$