ME 304 CONTROL SYSTEMS
Spring 2004
Sections 01, 03, 04

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SOLUTIONS TO FINAL EXAMINATION

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Time Allowed: 180 minutes
Open Notes and Books
All questions are equally weighted

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PROBLEM 1:
Consider the velocity control of a mass $m=13$kg by a proportional controller of gain $K_p$ in [N/V]. Suppose that the transfer function of the velocity sensor that generates velocity measurements in electrical voltage is given as

$$H(s) = \frac{3}{7s + 5}$$

where the unit of $H(s)$ is [V.s/m]. It can be shown that the closed loop transfer function between the velocity $v(t)$ of the mass and the reference voltage input $r(t)$ for this control system is

$$G_{VR}(s) = \frac{K_p(7s + 5)}{91s^2 + 65s + 3K_p}$$

where the unit of $G_{VR}(s)$ is [m/V.s].

a) Draw a detailed block diagram of this control system,
- defining and indicating all physical variables and
- entering an appropriate transfer function expression inside each block.

b) Verify the closed loop transfer function $G_{VR}(s)$ given above between the velocity $v(t)$ of the mass and the reference voltage input $r(t)$ for this control system by using your block diagram.

c) Note that for $K_p=11$, $G_{VR}(s)$ becomes

$$G_{VR}(s) = \frac{77s + 55}{91s^2 + 65s + 33}$$

Obtain the numerical values of the following parameters of this feedback system.
- Characteristic time $T (=1/\omega_n)$,
- Damping ratio $\zeta$,
- Numerator time constant $T_0$,
- Steady state gain $K$.

SOLUTION:

a) Detailed block diagram of the system

b) Closed loop transfer function can be obtained as
\[ G_{VR}(s) = \frac{K_p \frac{1}{13s}}{1 + K_p \frac{1}{13s} \frac{3}{7s + 5}} = \frac{K_p \frac{1}{13s}}{(13s)(7s + 5) + 3K_p} = \frac{K_p (7s + 5)}{91s^2 + 65s + 3K_p} \]

c) Normalized transfer function is obtained as

\[ G_{VR}(s) = \frac{11(7s + 5)}{91s^2 + 65s + 33} = \frac{55}{33} \left( \frac{7s + 1}{5} + \frac{5}{5} \right) = \frac{(5/3) \left( \frac{7s + 1}{5} \right)}{91s^2 + 65s + 33 + 1} = \frac{K(T_0s + 1)}{T^2s^2 + 2\zeta Ts + 1} \]

Hence,

i) Characteristic time \( T = \sqrt{\frac{91}{33}} \approx 1.66 \) [s],

ii) Damping ratio \( \zeta = \frac{65}{33}/2T \approx \frac{65}{66 \times 1.66} \approx 0.593 \) [nondimensional],

iii) Numerator time constant \( T_0 = \frac{7}{5} \approx 1.4 \) [s],

iv) Steady state gain \( K = \frac{5}{3} \approx 1.67 \) [m/V.s].
**PROBLEM 2:**

Consider the following block diagram representing the thermal model of a room.

\[ T(s) : \text{Laplace transform of the room temperature} \]
\[ U(s) : \text{Laplace transform of the heat flow rate} \]
\[ \text{supplied by a heater} \]
\[ To(s) : \text{Laplace transform of the ambient} \]
\[ \text{temperature} \]
\[ Tr(s) : \text{Laplace transform of the desired room} \]
\[ \text{temperature (not shown in the figure)} \]

a) Design an open-loop control system such that both

i) effect of ambient temperature is completely eliminated,

ii) room temperature will be at the desired level at steady-state, using controllers like

\[ u = A \]

where \( A \) is a constant to be determined. Assume that only one temperature sensor with the following first-order transfer function is available which is used for measuring the ambient temperature.

\[ G_s(s) = \frac{1}{s+1} \]

b) Draw the block diagram representation of the complete system showing the plant, controller and sensor clearly. You should indicate numerical values of all parameters in your block diagram.

**SOLUTION:**

a) \[ G_d = -\frac{3}{10s+1} \quad \text{and} \quad G_u = \frac{4}{10s+1} \Rightarrow G_d' = -\frac{3}{4} = -0.75 \quad \text{and} \quad G_u = \frac{10s+1}{4} \]

Since steady-state behavior is asked, the DC gains should be used in the open-loop controller transfer functions. Note that, the DC gain of the sensor is 1.

\[ G_d' = -\frac{3}{4} = -0.75 \quad \text{and} \quad G_u = \frac{1}{4} = 0.25 \]

b) Block Diagram Representation
PROBLEM 3:

In order to estimate the dynamic characteristics of a third order plant, it is decided to perform an open-loop impulse test. For this purpose, the plant input is applied as a unit impulse function, i.e. \( u(t) = \delta(t) \), and the plant output, \( c(t) \), is recorded as shown in the following figure. As the record is examined, it is particularly noted that the initial slope of the response curve appears very clearly to be zero. Referring to this figure, the input-output transfer function of the plant between \( c(t) \) and \( u(t) \) is estimated to be what is written on the right side of the figure.

\[
G_p(s) = \frac{1.25}{s(Ts + 1)^2}
\]

The following questions are related to the five features that make this transfer function a valid estimate. Answer these questions immediately below them with neat and clear explanations.

i) The first pole of \( G_p(s) \) is estimated to be \( p_1 = 0 \). Why?

After applying the impulsive input, the system settles down to a new equilibrium position different from the original one. Therefore, it is marginally stable with a pole at the origin of the complex plain. In other words, \( p_1 = 0 \). This fact is reflected by the free "s" factor in the denominator of \( G_p(s) \).

ii) The other two poles \( (p_2, p_3) \) of \( G_p(s) \) are estimated to be real and negative. Why?

There are no oscillations. Therefore, the remaining two poles must be real. This fact is reflected by two real poles, which happen to be \( p_2 = p_3 = -1/T \).

iii) \( G_p(s) \) is estimated to be without any zero. Why?

The response curve starts with zero slope. Therefore, \( T_0 = 0 \). That is, there is no zero or numerator dynamics. This fact is reflected by a constant numerator.

iv) The steady state gain of \( G_p(s) \) is estimated to be 1.25. Why?

The final value of the output is seen to be 1.25. The final value theorem also yields 1.25 as applied to \( G_p(s) \). Therefore, the steady state gain is correctly set to 1.25.

v) The nonzero real poles of \( G_p(s) \) are estimated to be very close to each other and therefore they are approximated to be equal. Why?

The fact that the initial slope appears very clearly to be zero implies that \( T_1 \approx T_2 \approx T \). Otherwise, if \( T_2 \) were dominantly larger than \( T_1 \), then the initial zero slope would disappear very quickly and the response curve would look almost like that of a simpler system with a lower order characteristic polynomial such as \( D'(s) = s(T_2s + 1) \). Therefore, it is justified to take \( T_1 = T_2 = T \).
PROBLEM 4:
Consider a plant with the following transfer function between its output $c(t)$ and its controlling input $u(t)$:

$$G_p(s) = \frac{125}{s(s+10)^2}.$$ 

This plant is to be controlled by installing a unity feedback proportional controller with a gain $K$.

a) Show that the characteristic polynomial of the closed loop system will be

$$D(s) = s^3 + 20s^2 + 100s + 125K.$$ 

b) In addition to guaranteeing stability of the closed loop system, it is also required that $\varepsilon_{ss} < 0.1$, where $\varepsilon_{ss}$ is the steady state error that occurs if the reference input is a unit ramp function, i.e. $r(t) = th(t)$. Show that these requirements are fulfilled if $8 < K < 16$.

c) Show that there exists a value of $K$ within the range indicated in Part (a) and find that value, which makes $\mu_s = 1$, where $\mu_s$ is the stability margin of the closed loop system.

SOLUTION:

a) OLTF and ERTF:

$$G_o(s) = KG_p(s) = \frac{125K}{s(s+10)^2}, \quad G_e(s) = \frac{1}{1+G_o(s)} = \frac{s(s+10)^2}{s^3 + 20s^2 + 100s + 125K}.$$ 

Characteristic Polynomial: 

$$D(s) = s^3 + 20s^2 + 100s + 125K.$$ 

Routh Criterion for 3rd order systems: 

$$20 \times 100 > 1 \times 125K \quad \rightarrow \quad K < 16.$$ 

Steady state error for $R(s) = 1/s^2$: 

$$\varepsilon_{ss} = \frac{100}{125K} < 0.1 \quad \rightarrow \quad K > 8.$$ 

Combined condition: 

$$8 < K < 16. \quad \text{(QED)}$$ 

b) Characteristic polynomial with shifted argument:

$$\bar{D}(z) = D(z-1) = (z-1)^3 + 20(z-1)^2 + 100(z-1) + 125K,$$

$$\bar{D}(z) = z^3 + 17z^2 + 63z + (125K - 81).$$

There are two possibilities for $\bar{D}(z)$ to have a root or roots with zero real part:

(i) $125K - 81 = 0 \quad \rightarrow \quad K = 0.648$ (This value is not valid; it is out of range)

(ii) $17 \times 63 = 1 \times (125K - 81) \quad \rightarrow \quad K = 9.216$ (This is the valid value)
**PROBLEM 5:**
Consider the step response of an underdamped second-order system with no numerator dynamics. It is desired to have the following requirements on the transient behavior of the system.
- The settling time of the system is between 0.5 and 1 second according to 2% criterion.
- The maximum overshoot of the system is between 1% and 10%.

a) Show the region on the complex s-plane satisfying both of these requirements.

b) Determine and indicate the special pole locations on your plot corresponding to each of the following specifications while satisfying the above requirements.
   i) minimum peak time,
   ii) maximum undamped natural frequency,
   iii) minimum overshoot and minimum settling time.
   You may assign letters to the pole locations corresponding to each of the above parts as i, ii, or iii. Explain your answers clearly.

c) Determine the corresponding settling time and maximum overshoot for each of the above cases i, ii and iii.

**SOLUTION:**

\[
0.5 \leq t_s \leq 1 \quad (2\% \text{ settling criterion})
\]

\[
t_s = \frac{4}{\zeta \omega_n}
\]

\(t_s=1 \Rightarrow \zeta \omega_n=4\) and \(t_s=0.5 \Rightarrow \zeta \omega_n=8\)

\[
0.01 \leq M_p \leq 0.1
\]

\[
M_p = \exp \left[ -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right] \times 100 \% \quad \text{and} \quad \cos \beta = \zeta
\]

\(\varepsilon_p=0.01 \Rightarrow \zeta=0.83 \quad (\beta=34.3^\circ)\) and \(\varepsilon_p=0.1 \Rightarrow \zeta=0.59 \quad (\beta=53.8^\circ)\)

\(\zeta \omega_n=8, \beta=34.3^\circ, \zeta=0.83 \Rightarrow \omega_d=5.5 \text{ rad/s} \Rightarrow p_{1,2}=-8 \pm j5.5\)

i) Minimum peak time \(\Rightarrow\) Point with maximum \(\omega_d\) value

\[
\omega_d = \omega_n \sqrt{1-\zeta^2}
\]

\(\zeta \omega_n=8, \beta=53.8^\circ, \zeta=0.59 \Rightarrow \omega_d=10.9 \text{ rad/s} \Rightarrow p_{1,2}=-8 \pm j10.9\)

\(t_s=0.5 \text{ s and } M_p=10 \%\)

ii) Maximum undamped natural frequency \(\Rightarrow\) Circle with maximum radius (\(\omega_n\)) value

\(\zeta \omega_n=8, \beta=53.8^\circ, \zeta=0.59 \Rightarrow \omega_d=10.9 \text{ rad/s} \Rightarrow p_{1,2}=-8 \pm j10.9\)

\(t_s=0.5 \text{ s and } M_p=10 \%\)

iii) Minimum overshoot and minimum settling time \(\Rightarrow\) Point with minimum \(\beta\) and maximum \(\sigma\) value

\(t_s=0.5 \text{ s and } M_p=1 \%\)

Half of the complex s-plane
PROBLEM 6:
The transfer function of a system is given as
\[ G(s) = \frac{50(s^2 + 10s + 100)}{s^2(s + 500)} \]

a) Draw the magnitude plot of its Bode diagram below using straight line approximation.

b) For the same system, determine the steady state amplitude of the output analytically using the transfer function above for a sinusoidal input of \( x(t) = (5 \sin(10t + 0.57)) \delta(t) \).

SOLUTION:

a) Normalization of the transfer function gives
\[ G(s) = \frac{50(s^2 + 10s + 100)}{s^2(s + 500)} = \frac{5000(0.01s^2 + 0.1s + 1)}{500s^2(0.002s + 1)} = \frac{G_1}{10(0.01s^2 + 0.1s + 1)} = \frac{G_2}{s^2(0.002s + 1)} = \frac{G_3}{G_4} \]

b) For \( M(\omega) = |G(j\omega)| = \frac{50[(j\omega)^2 + 10j\omega + 100]}{|(j\omega)^2|(|j\omega + 500|)} = \frac{50}{\omega^2|j\omega + 500|} \approx \frac{5000}{50000} = 0.1 \]

Hence the steady state amplitude of the output becomes \( 0.1*5 = 0.5 \).