PROBLEM 1

Consider the mechanism shown in the figure. The dimensions of the mechanism are,

\[ |AC| = 80 \text{ mm}, \quad |AF| = 150 \text{ mm}, \quad |BC| = |CD| = |DB| = 50 \text{ mm}, \quad |DE| = 90 \text{ mm} \]

a) If the input is \( \theta_{12} = 30^\circ \), find all possible solutions (corresponding to all closures of the mechanism) for the unknown joint variables using a graphical approach and present your results in a tabulated form. (You are expected to use a drawing tool)

b) Write down the loop closure equations and find the values of the unknown joint variables analytically if the input is \( \theta_{12} = 30^\circ \).
a) The unknown joint variables can be obtained using a graphical approach as follows, 

<table>
<thead>
<tr>
<th>Closure</th>
<th>$s_{32}$ [mm]</th>
<th>$\theta_{14}$ [°]</th>
<th>$\theta_{15}$ [°]</th>
<th>$s_{16}$ [mm]</th>
<th>$\theta_{15}$ [°]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closure 1</td>
<td>39.28</td>
<td>156.87</td>
<td>96.87</td>
<td>97.88</td>
<td>32.41</td>
</tr>
<tr>
<td>Closure 2</td>
<td>39.28</td>
<td>156.87</td>
<td>96.87</td>
<td>1.40</td>
<td>327.59</td>
</tr>
<tr>
<td>Closure 3</td>
<td>99.28</td>
<td>83.13</td>
<td>23.13</td>
<td>106.38</td>
<td>74.52</td>
</tr>
<tr>
<td>Closure 4</td>
<td>99.28</td>
<td>83.13</td>
<td>23.13</td>
<td>-67.09</td>
<td>285.48</td>
</tr>
</tbody>
</table>
b) Firstly, determine the number of independent loops. The mechanism has 6 links and 7 joints. Therefore,
\[ L = j - l + 1 = 7 - 6 + 1 = 2 \]

Then, the loop closure equations are
\[ \overline{AB} = \overline{AC} + \overline{CB} \quad \Rightarrow \quad s_{32} e^{i\theta_1} = |AC| + |CB| e^{i\theta_4} \quad (1) \]
\[ \overline{CD} + \overline{DE} = \overline{CF} + \overline{FE} \quad \Rightarrow \quad |CD| e^{i\theta_1} + |DE| e^{i\theta_5} = |CF| + i\alpha_6 \quad (2) \]

Now consider Eq. (1), separating the real and complex parts gives,
\[ \begin{align*}
    s_{32} \cos \theta_1 & = |AC| + |CB| \cos \theta_4^* \\
    s_{32} \sin \theta_1 & = |CB| \sin \theta_4^*
\end{align*} \quad (3) \]
\[ \begin{align*}
    (s_{32} \cos \theta_1 - |AC|)^2 & = (|CB| \cos \theta_4^*)^2 \\
    (s_{32} \sin \theta_1)^2 & = (|CB| \sin \theta_4^*)^2
\end{align*} \quad (4) \]

First, \( \theta_4^* \) can be eliminated to obtain \( s_{32} \),
\[ \begin{align*}
    (s_{32} \cos \theta_1)^2 - |AC|^2 s_{32} + |AC|^2 - |CB|^2 = 0 \\
    s_{32} = |AC| \cos \theta_1 + \sigma_1 \sqrt{|AC|^2 \cos \theta_1^2 - |AC|^2 + |CB|^2}
\end{align*} \]

where \( \sigma_1 = \pm 1 \) (two solutions)

- For \( |AC| = 80 \text{ mm}, |CB| = 50 \text{ mm} \) and \( \theta_1 = 30^\circ \),
  \[ s_{32} = 80 \cos 30^\circ + \sigma_1 \sqrt{(80 \cos 30^\circ)^2 - 80^2 + 50^2} \]

Therefore, \( s_{32} = 99.282 \text{ mm} \) for \( \sigma_1 = +1 \) and \( s_{32} = 39.282 \text{ mm} \) for \( \sigma_1 = -1 \).

From Eq. (3) and (4),
\[ \begin{align*}
    \sin \theta_4^* & = s_{32} \sin \theta_1 / |CB| \quad \text{and} \quad \cos \theta_4^* = s_{32} \cos \theta_1 - |AC| / |CB|
\end{align*} \]

So, neglecting common positive denominator for “atan2()”,
\[ \theta_4^* = \text{atan2}(s_{32} \sin \theta_1, s_{32} \cos \theta_1 - |AC|) \]

- For \( |AC| = 80 \text{ mm}, \theta_1 = 30^\circ \) and \( s_{32} = 99.282 \text{ mm} \)
  \[ \theta_4^* = \text{atan2}(99.282 \sin 30^\circ, 99.282 \cos 30^\circ - 80) = 83.13^\circ \quad \Rightarrow \quad \theta_4 = \theta_4^* - 60^\circ = 23.13^\circ \]

- For \( |AC| = 80 \text{ mm}, \theta_1 = 30^\circ \) and \( s_{32} = 39.282 \text{ mm} \)
  \[ \theta_4^* = \text{atan2}(39.282 \sin 30^\circ, 39.282 \cos 30^\circ - 80) = 156.87^\circ \quad \Rightarrow \quad \theta_4 = \theta_4^* - 60^\circ = 96.87^\circ \]

Now consider Eq. (2), separating the real and complex parts gives,
\[ |CD| \cos \theta_4 + |DE| \cos \theta_5 = |CF| \]
(5)
\[ |CD| \sin \theta_4 + |DE| \sin \theta_5 = s_6 \quad (6) \]

First, \( \theta_5 \) can be eliminated to obtain \( s_6 \),

\[
\left( |CF| - |CD| \cos \theta_4 \right)^2 = \left( |DE| \cos \theta_5 \right)^2 \\
\left( s_6 - |CD| \sin \theta_4 \right)^2 = \left( |DE| \sin \theta_5 \right)^2
\]

Summation of the both sides and expanding the squares,

\[
s_6^2 - 2|CD| \sin \theta_4 s_6 + |CF|^2 + |CD|^2 - |DE|^2 - 2|CF||CD| \cos \theta_4 = 0
\]

So, \( s_6 \) is found as follows,

\[
s_6 = |CD| \sin \theta_4 + \sigma_2 \sqrt{\left( |CD| \sin \theta_4 \right)^2 - |CF|^2 - |CD|^2 + |DE|^2 + 2|CF||CD| \cos \theta_4}
\]

where \( \sigma_2 = \pm 1 \) (two solutions)

- For, \( |CD| = 50 \text{ mm}, \ |DE| = 90 \text{ mm}, \ |CF| = 70 \text{ mm} \) and \( \theta_4 = 23.13^\circ \),

\[
s_6 = 50 \cdot \sin 23.13^\circ + \sigma_2 \sqrt{\left( 50 \cdot \sin 23.13^\circ \right)^2 - 70^2 - 50^2 + 90^2 + 2 \cdot 70 \cdot 50 \cdot \cos 23.13^\circ}
\]

Therefore, \( s_6 = 106.38 \text{ mm for } \sigma_2 = +1 \) and \( s_6 = -67.09 \text{ mm for } \sigma_2 = -1 \).

- For, \( |CD| = 50 \text{ mm}, \ |DE| = 90 \text{ mm}, \ |CF| = 70 \text{ mm} \) and \( \theta_4 = 96.87^\circ \),

\[
s_6 = 50 \cdot \sin 96.87^\circ + \sigma_2 \sqrt{\left( 50 \cdot \sin 96.87^\circ \right)^2 - 70^2 - 50^2 + 90^2 + 2 \cdot 70 \cdot 50 \cdot \cos 96.87^\circ}
\]

So, \( s_6 = 97.88 \text{ mm for } \sigma_2 = +1 \) and \( s_6 = 1.40 \text{ mm for } \sigma_2 = -1 \).

From Eq. (5) and (6),

\[
\sin \theta_5 = \frac{s_6 - |CD| \sin \theta_4}{|DE|} \quad \text{and} \quad \cos \theta_5 = \frac{|CF| - |CD| \cos \theta_4}{|DE|}
\]

So, neglecting common positive denominator for “atan2()”

\[
\theta_5 = \text{atan2}\left( s_6 - |CD| \sin \theta_4, |CF| - |CD| \cos \theta_4 \right)
\]

- For, \( |CD| = 50 \text{ mm}, \ |CF| = 70 \text{ mm}, \ |DE| = 23.13^\circ \) and \( s_6 = 106.38 \text{ mm} \),

\[
\theta_5 = \text{atan2}\left( 106.38 - 50 \cdot \sin (23.13^\circ), 70 - 50 \cdot \cos (23.13^\circ) \right) = 74.52^\circ
\]

- For, \( |CD| = 50 \text{ mm}, \ |CF| = 70 \text{ mm}, \ |DE| = 23.13^\circ \) and \( s_6 = -67.09 \text{ mm} \),

\[
\theta_5 = \text{atan2}\left( -67.09 - 50 \cdot \sin (23.13^\circ), 70 - 50 \cdot \cos (23.13^\circ) \right) = 285.48^\circ
\]

- For, \( |CD| = 50 \text{ mm}, \ |CF| = 70 \text{ mm}, \ |DE| = 96.87^\circ \) and \( s_6 = 97.88 \text{ mm} \),

\[
\theta_5 = \text{atan2}\left( 97.88 - 50 \cdot \sin (96.87^\circ), 70 - 50 \cdot \cos (96.87^\circ) \right) = 32.41^\circ
\]

- For, \( |CD| = 50 \text{ mm}, \ |CF| = 70 \text{ mm}, \ |DE| = 96.87^\circ \) and \( s_6 = 1.40 \text{ mm} \),

\[
\theta_5 = \text{atan2}\left( 97.88 - 50 \cdot \sin (96.87^\circ), 70 - 50 \cdot \cos (96.87^\circ) \right) = 327.59^\circ
\]
PROBLEM 2
Consider the inverted slider crank mechanism shown in the figure.

The dimensions of the mechanism are given as

\[ |AC| = b_1 = 18 \text{ cm} \quad |AB| = b_2 = 8 \text{ cm} \quad |BE| = b_3 = 30 \text{ cm} \quad |CD| = b_4 = 5 \text{ cm} \]

a) Using an analytical approach, obtain the equations for the joint variables in terms of input \( \theta_{12} \) and the link lengths.

For the closure given in the figure,

b) implement the expressions found in part (a) into a computer program (Matlab, MathCAD, etc.) and perform full cycle position analysis for input \( \theta_{12} \) changing from 0 to 360 degrees with increments of at most 5 degrees.

c) plot the variations of \( s_{43} \) and \( \theta_{14} \) with respect to \( \theta_{12} \) for full crank rotation.

d) plot the curve traced by coupler point E.

Note: Make sure to submit the source code of your programs, explain the procedure using comments etc and fully annotate the plots. There is no need to submit the print-out of the tabulated results.

Hint: An example about using MATLAB for mechanism analysis is posted on the website.
SOLUTION

a) Virtually disconnect the mechanism at point B:

\[ b_2 e^{i\theta_2} = b_1 + b_4 e^{i\theta_4} + s_{43} e^{i(\theta_4 + \pi/2)} \]  

(1)

Rewrite the LCE:

\[ b_2 e^{i\theta_2} = b_1 + (b_4 + is_{43}) e^{i\theta_4} \]  

(2)

Separate the real and imaginary parts:

\[ b_2 \cos \theta_{12} = b_1 + (b_4 \cos \theta_{14} - s_{43} \sin \theta_{14}) \]  

(3)

\[ b_2 \sin \theta_{12} = b_4 \sin \theta_{14} + s_{43} \cos \theta_{14} \]  

(4)

Multiply Equation (3) by \( \cos \theta_{14} \) and Equation (4) by \( \sin \theta_{14} \) and add the equations to get

\[ (b_2 \cos \theta_{12} - b_1) \cos \theta_{14} + (b_2 \sin \theta_{12}) \sin \theta_{14} = b_4 \]  

(5)

This equation is in the form of

\[ A \cos \theta_{14} + B \sin \theta_{14} = C \]  

(6)

where,

\[ A = b_2 \cos \theta_{12} - b_1 \]

\[ B = b_2 \sin \theta_{12} \]

\[ C = b_4 \]  

(7)

Equation (6) can be solved using half tangent method.

\[ t_{14} = \tan \left( \frac{\theta_{14}}{2} \right), \quad \cos \theta_{14} = \frac{1 - t_{14}^2}{1 + t_{14}^2}, \quad \sin \theta_{14} = \frac{2t_{14}}{1 + t_{14}^2} \]

\[ (A + C) t_{14}^2 - 2Bt_{14} + (C - A) = 0 \]  

(8)

Then

\[ t_{14} = \frac{B + \sigma \sqrt{B^2 + A^2 - C^2}}{A + C}, \quad \sigma = \pm 1 \]

(9)

There are two solutions for \( \theta_{14} \), therefore the mechanism has two closures. Once the value of \( \theta_{14} \) is calculated, the remaining unknown joint variable \( s_{43} \) can be calculated using Equation (3) or (4). Considering Equation (3), \( s_{43} \) can be written as

\[ s_{43} = \frac{(b_2 \cos \theta_{12} - b_1 - b_2 \cos \theta_{14})}{\sin \theta_{14}} \]  

(10)
b) The solution is implemented into a Matlab m-file with the following script.

```matlab
% Me 301 Theory of Machines
% Fal 2008 Hw 3 Solution
% Position Analysis and Visual Simulation of Inverted Slider Crank Mechanism

clear all;clc;close all;

% link lengths
b1 = 18;
b2 = 8;
b3 = 30;
b4 = 5;

% input joint variable changing from
theta12 = (0:1:360)*pi/180;
sigma = -1; % closure selection

% A cos(theta) + B sin(theta) = C
A = b2 * cos(theta12) - b1;
B = b2 * sin(theta12);
C = b4;

% Solution of the nonlinear equation via Half Tangents Method
theta14 = atan2(2*t14,1-t14.^2);
s43 = -(b2 * cos(theta12) - b1 - b4*cos(theta14))./sin(theta14);

% Calculation of the position of the points A, B, C, D, E
xA = zeros(size(theta12)); yA = zeros(size(theta12));
xB = xA + b2*cos(theta12); yB = yA + b2*sin(theta12);
xC = xA + b1; yC = yA;
xD = xC + b4*cos(theta14); yD = yC + b4*sin(theta14);
xE = xB + b3*cos(theta14 + sigma*pi/2); yE = yB + b3*sin(theta14 + sigma*pi/2);

% this part is for visualizing the slider
% model slider with inputs swidth, slength, sangle and x&y positions
scentX, scentY
swidth = 2;
slength = 2;
sangle = theta14;
scentX = xD;
scentY = yD;

% this part calculates the corners of the slider for each value of the
% input, there is no need to modify this part
sxA = scentX + swidth*cos(sangle+pi/2)/2 + slength*cos(sangle)/2;
syA = scentY + swidth*sin(sangle+pi/2)/2 + slength*sin(sangle)/2;
sxB = sxA + swidth*cos(sangle-pi/2);
syB = syA + swidth*sin(sangle-pi/2);
sxC = sxB + slength*cos(sangle-pi);
syC = syB + slength*sin(sangle-pi);
sxD = sxC + width*cos(sangle+pi/2);
syD = syC + width*sin(sangle+pi/2);
sliderX = [sxA; sxB; sxC; sxD; sxA]';
sliderY = [syA; syB; syC; syD; syA]';

% simulating the motion of the mechanism
set(gca,'NextPlot','replacechildren','DataAspectRatio',[1 1 1]);
grid on;
axis([-20 40 -20 40]);

for i=1:size(theta12,2)
    plot(x(i,:),y(i,:),'r-o',sliderX(i,:),sliderY(i,:),'k-',xE(1:i),yE(1:i),'b--',
    'LineWidth',2);
```
F(i) = getframe;
end

% movie(F,2,36)  % repeat the simulation twice with 36 frame per second

% plots
figure
plot(xE,yE);    % point E's curve
title('The Curve Traced by Coupler Point E');
xlabel('x [cm]');
ylabel('y [cm]');

figure
plot(theta12*180/pi,s43);
title('$s_{43}$ vs $\theta_{12}$');
xlabel('$\theta_{12}$ [deg]');
ylabel('$s_{43}$ [cm]');

figure
plot(theta12*180/pi,theta14*180/pi);
title('$\theta_{14}$ vs $\theta_{12}$');
xlabel('$\theta_{12}$ [deg]');
ylabel('$\theta_{14}$ [deg]');

c)

d)