You should submit your homework on its due time (17:00). No extension is given afterward.

Problem 1:
Consider a unity feedback control system with the following plant transfer function:

\[ G_p(s) = \frac{4}{s^2(s+2)} = \frac{C(s)}{U(s)} \]

Recall that, in a unity feedback control system, the sensor is perfect. That is, it is noise-free and its transfer function is \( H(s) = 1 \).

The plant is to be controlled with a PID controller. In other words, the transfer function of the controller is:

\[ G_c(s) = K_p + \frac{K_i}{s} + K_ds = \frac{U(s)}{E(s)} \]

a) For the control system described above, derive the closed-loop transfer function \( M(s) = M_{cr}(s) \) between the output \( C(s) \) and the reference input \( R(s) \).

b) According to the Routh-Hurwitz criterion, derive the inequality conditions on \( K_p \), \( K_i \), and \( K_d \) for the stability of the closed-loop system.

c) Show that it is possible to select the derivative gain as \( K_d = 8 \). Then, selecting the relevant mid-range values, find the proportional gain \( K_p \) and the integral gain \( K_i \).

d) Recall that a PID controller can also be parameterized by an amplification gain \( K_a \) and the derivative and integral action times \( T_d \) and \( T_i \). Determine these alternative parameters corresponding to the gains determined in Part (c).
**SOLUTION:**

a) The input-output equation can be written as follows:

\[ C(s) = G_c(s)G_p(s)[R(s) - C(s)] \]  \hspace{1cm} (1)

Rearrange Eq. (1) to obtain the following closed-loop transfer function:

\[ M(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{G_c(s)G_p(s) + 1} \]  \hspace{1cm} (2)

The transfer function of the PID controller is:

\[ G_c(s) = K_p + \frac{K_i}{s} + K_ds \]  \hspace{1cm} (3)

Substitute the plant and controller transfer functions into Eq. (2) so that

\[ M(s) = \frac{(K_p + \frac{K_i}{s} + K_ds)\left(\frac{4}{s^2(s + 2)}\right)}{(K_p + \frac{K_i}{s} + K_ds)\left(\frac{4}{s^2(s + 2)}\right) + 1} \]  \hspace{1cm} (4)

Rearrange \( M(s) \) into the following regular form:

\[ M(s) = \frac{4K_ds^2 + 4K_ps + 4K_i}{s^4 + 2s^3 + 4K_ds^2 + 4K_ps + 4K_i} \]  \hspace{1cm} (5)

b) The characteristic polynomial, which is the denominator of \( M(s) \), is identified as follows:

\[ Den[M(s)] = D(s) = s^4 + 2s^3 + 4K_ds^2 + 4K_ps + 4K_i \]  \hspace{1cm} (6)

As noted, \( D(s) \) passes the Hurwitz test, i.e., all its coefficients are strictly positive. In that case, the conditions of stability must be determined according to the Routh Criterion. For this purpose, the Routh Array is constructed as shown below:

<table>
<thead>
<tr>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s )</th>
<th>( 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>4K_d</td>
<td>4K_i</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4K_p</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( c_2 )</td>
<td>( d_2 = 4K_i )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( c_1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>( c_0 = 4K_i )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
According to the Hurwitz criterion, the coefficients of the powers of \( s \) in \( D(s) \) should be strictly positive. Actually, this requirement is automatically satisfied because the control gains are already positive. That is,

\[
K_d > 0 \quad (\text{Condition 1})
\]
\[
K_p > 0 \quad (\text{Condition 2})
\]
\[
K_i > 0 \quad (\text{Condition 3})
\]

As additional requirements according to the Routh criterion, the elements of the first column of the Routh Array (RA) must also be positive. The unknown entries of the RA are found as follows:

\[
c_2 = \frac{8K_d - 4K_p}{2}
\]
\[
c_1 = \frac{4c_2K_p - 8K_i}{c_2} = \frac{16K_pK_d - 8K_p^2 - 8K_i}{4K_d - 2K_p}
\]

As a result, the 4th and 5th conditions come out as:

\[
K_p < 2K_d \quad (\text{Condition 4})
\]
\[
K_p(2K_d - K_p) - K_i > 0 \quad (\text{Condition 5})
\]

c) The derivative gain of the PID controller is suggested to be taken as \( K_d = 8 \) in the question. Hence, the conditions on the other controller gains become:

\[
0 < K_p < 16
\]
\[
0 < K_i < K_p(16 - K_p)
\]

It is also suggested to pick the mid-range values for the other control gains. So, they are found as follows from the above conditions:

\[
K_p = 8
\]
\[
K_i = 32
\]

d) In order to find the amplifier gain \( K_a \) and the time constants \( T_i \) and \( T_d \) of the integral and derivative actions, the transfer function of the controller can be expressed in the following two equivalent forms:

\[
G_c(s) = \left( K_p + \frac{K_i}{s} + K_ds \right) = K_a \left( 1 + \frac{1}{T_i s} + T_ds \right)
\]
With the gains found above, equation (7) becomes:

\[
\left( 8 + \frac{32}{s} + 8s \right) = 8 \left( 1 + \frac{1}{T_i s} + T_d s \right)
\]

Therefore, the unknown parameters can be found as follows:

\[
K_a = 8 \\
T_i = \frac{8}{32} = 0.25 \\
T_d = \frac{8}{8} = 1
\]