IMPORTANT NOTE: This homework is composed of supplementary problems. Solutions will NOT be collected and graded.

Question 1:

The two systems in the figures (a) and (b) are equivalent, such that the vertical motion of point A in figure (a) is the same as the motion of the mass in figure (b). Find the equivalent mass, the equivalent damping constant, the equivalent spring constant and the equivalent force as shown in figure (b). Assume small oscillations.

\[ m_{eq} = m + \frac{I_G}{36L^2}, \quad c_{eq} = \frac{c}{9}, \quad k_{eq} = \frac{k}{4}, \quad F_{eq} = \frac{2}{3} F(t) \]

Question 2:

The system shown below is a model of an automobile clutch. Find the equivalent system. Assume small displacements measured from static equilibrium position.
Solution:

Finding equivalent inertia:

\[(KE): \quad \frac{1}{2} \cdot m_{\text{eq}} \cdot \dot{x}_E^2 = \frac{1}{2} \cdot m_1 \cdot \dot{x}_E^2 + \frac{1}{2} \cdot m_E \cdot \dot{x}_E^2\]

\[\Rightarrow \quad m_{\text{eq}} = m_1 + m_E \quad \text{EQUA}_\text{I}\]

Finding equivalent stiffness:

\[(PE): \quad \frac{1}{2} \cdot k_{\text{eq}} \cdot x_E^2 = \frac{1}{2} \cdot k_1 \cdot \left(\frac{x_E}{2a}\right)^2 + 2 \cdot \frac{1}{2} \cdot k_2 \cdot x_E^2 + m_1 \cdot g \cdot 2a(\cos(\frac{x_E}{2a}) - 1)\]

\[
\text{SINCE } \cos(\alpha) \text{ can be written in Taylor series expansion as } \cos(\alpha) = 1 - \frac{\alpha^2}{2} + ... \]

\[
\text{THEN } \cos(\frac{x_E}{2a}) \approx 1 - \frac{\left(\frac{x_E}{2a}\right)^2}{2} \approx 1 - \frac{x_E^2}{8a^2} \]

Substituting this into (PE) equation and solving for \(k_{\text{eq}}\) yields

\[\Rightarrow \quad k_{\text{eq}} = \frac{1}{4} \cdot k_1 + 2k_2 - \frac{m_1 \cdot g}{2a} \quad \text{EQUA}_\text{II}\]
Finding equivalent damping:

$$\theta = \frac{x_E}{2 \cdot a}, \quad \dot{\theta} = \frac{\dot{x}_E}{2 \cdot a}, \quad \delta \theta = \frac{\delta x_E}{2 \cdot a}$$

$$(\delta U_{\text{disp}}): \quad -c_{eq} \cdot \dot{x}_E \cdot \delta x_E = -c_t \cdot \dot{\theta} \cdot \delta \theta - c \cdot x_E \cdot \delta x_E$$

$$\Rightarrow \quad -c_{eq} \cdot \dot{x}_E \cdot \delta x = -c_t \cdot \frac{\dot{x}_E}{2 \cdot a} \cdot \frac{\delta x_E}{2 \cdot a} - c \cdot x_E \cdot \delta x_E = -\left(\frac{c_t}{4 \cdot a^2} + c\right) \cdot \dot{x}_E \cdot \delta x_E$$

$$\Rightarrow \quad c_{eq} = \frac{c_t}{4 \cdot a^2} + c \quad \text{EQUA}_{III}$$

Finding equivalent forcing:

$$(\delta U_{\text{ext}}): \quad F_{eq} \cdot \delta x_E = 0 \cdot \delta x_E$$

$$\Rightarrow \quad F_{eq} = 0 \quad \text{EQUA}_{IV}$$

Then, for the given system, equation of motion may be written by using EQUA I, II, III and IV as:

$$(m_1 + m_E) \cdot \ddot{x}_E + \left(\frac{c_t}{4 \cdot a^2} + c\right) \cdot \dot{x}_E + \left(\frac{1}{4} k_1 + 2 k_2 - \frac{m_1 \cdot g}{2a}\right) \cdot x_E = 0$$
Question 3:

The cylinders are rolling on each other without slipping. Obtain the equation of motion in terms of the mass position \( x \) by using the energy method. The beams are rigid and massless. Assume all movements to be small.

**Solution:**
The equation of motion for this system can be found by using the energy method in terms of the position of the mass $x$.

The energy method: \[ \frac{d}{dt}(KE + PE) = P_m \]

The kinetic energy of the system,

\[ KE = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_1 \dot{\omega}_1^2 + \frac{1}{2} J_2 \dot{\omega}_2^2 \]

The kinematic equations for $\omega_1$, $\omega_2$

\[ \omega_1 = \frac{x_R}{r_1}, \quad \omega_2 = \frac{x_R}{r_2}, \quad x_R = \frac{x}{2}, \quad \dot{x}_R = \frac{\dot{x}}{2} \]

So,

\[ \omega_1 = \frac{\dot{x}}{2r_1}, \quad \omega_2 = \frac{\dot{x}}{2r_2} \]

\[ KE = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_1 \left(\frac{\dot{x}}{2r_1}\right)^2 + \frac{1}{2} J_2 \left(\frac{\dot{x}}{2r_2}\right)^2 \]

\[ KE = \frac{1}{2} m \dot{x}^2 + \frac{1}{8r_1^2} J_1 \dot{x}^2 + \frac{1}{8r_2^2} J_2 \dot{x}^2 \]

The potential energy of the system,

\[ PE = \frac{1}{2} K x_k^2 \]

Where $x_k = \frac{x_R}{2} = \frac{x}{4}$

\[ PE = \frac{1}{32} K x^2 \]

The potential energy of $mg$ is not taken into account in potential energy equation because $mg$ in this system has not a restoring effect and $x$ is measured from static equilibrium position.

\[ P_m = T(t)\omega_1 - B(\dot{x}_B)^2 \]

where $x_B = \frac{x}{2}, \quad \dot{x}_B = \frac{\dot{x}}{2}$

\[ P_m = T(t)\left(\frac{\dot{x}}{2r_1}\right) - B\left(\frac{\dot{x}}{2}\right)^2 \]
Question 4:

Write the equation of motion of the given system in terms of $x$.

Answer: \[ m_{eq} = \frac{3m_c}{2} + \frac{m_1}{3} + m_2 + \left(1 + \frac{r_p^2}{\ell_1^2}\right) m + \frac{I_p}{\ell_1^2}, \quad k_{eq} = k_2 + m_1 g + k_i \frac{r_p^2}{\ell_1^2} \]
Question 5:

The system shown in the figure is to be represented by an equivalent single-degree-of-freedom model with $x_A$ as the generalized coordinate. Calculate the equivalent spring constant and the equivalent mass. The numerical data are as follows: $E = 200 \text{ GPa}$, $I = 180 \text{ cm}^4$, $m_b = 40 \text{ kg}$, $m_c = 50 \text{ kg}$, $a = 0.8 \text{ m}$, $b = 0.4 \text{ m}$, $L = 1.2 \text{ m}$. The beam deflection formula for this case is given below.

$$y(x) = \begin{cases} \frac{Fbx}{6EIL}(L^2 - x^2 - b^2), & 0 < x < a \\ \frac{Fb}{6EIL} \left[ \frac{L}{b} (x-a)^3 + (L^2 - b^2)(x-x^3) \right], & a < x < L \end{cases}$$

Answer: $m_{eq} = 73.4 \text{ kg}$, $k_{eq} = 12.7 \text{ kN/mm}$
Question 6:

The system shown is to be represented by an equivalent linear single d.o.f. model with $\theta$ as the generalized coordinate.

At static equilibrium the rigid bar $AB$ is vertical. The oscillations are small and the ropes are massless, inextensible and always in tension. The uniform beam $DE$ is fixed at both ends. Its modulus of elasticity is $E$ and cross-sectional area moment of inertia is $I$.

(a) Find the equivalent torsional spring constant, $k_{eq}$.

(b) Find the equivalent torsional damping constant, $c_{eq}$.

(c) Find the equivalent external torque, $T_{eq}$.

(d) Derive an expression for the percentage of beam mass to be lumped at $C$.

(e) Find the equivalent inertia, $I_{eq}$.

Solution:

\[
y(\frac{L}{2}) = \frac{F L^2}{48EI} (3L - 2L) = \frac{FL^3}{192EI} \Rightarrow k_b = \frac{F}{y} = \frac{192EI}{L^3}
\]

\[
V = \frac{1}{2} k_b (a+b)^2 \theta^2 + \frac{1}{2} k ((a+b) \theta + a \theta)^2 + m_3 g \frac{a+b}{2} (1 - \cos(\theta)) = \frac{1}{2} k_{eq} \theta^2
\]
Note that
\[ (1 - \cos(\theta)) = \frac{\theta^2}{2} \]

Thus
\[ k_{eq} = \frac{192EI}{L^3} (a+b)^2 + k(2a+b)^2 + m_g \frac{a+b}{2} \]

b) 
\[ P_e = -c_{eq} \dot{\theta}^2 = -c_1 (2a+b)^2 \dot{\theta}^2 - c_2 a^2 \dot{\theta}^2 \]
\[ c_{eq} = c_1 (2a+b)^2 + c_2 a^2 \]

c) 
\[ P_e = T_{eq} \dot{\theta} = (T_0 \sin(\omega t)) \frac{a \dot{\theta}}{r_1} \Rightarrow T_{eq} = \frac{a}{r_1} T_0 \sin(\omega t) \]

d) 
\[ y(z) = cz^2 \left(3L_2 - 4z\right) \]
\[ y \left( \frac{L_2}{2} \right) = y_e \Rightarrow C = \frac{4y_e}{L_2^3} \]
\[ y(z) = y_e \frac{4}{L_2} z^2 \left(3L_2 - 4z\right) \]
\[ \dot{y}(z) = \dot{y}_e \frac{4}{L_2} z^2 \left(3L_2 - 4z\right) \]
\[ T_b = \frac{1}{2} \int_0^{L_2/2} \dot{y}^2 \, dm \quad \Rightarrow \quad dm = \frac{m_b}{L_2} \, dz \]
\[ T_b = \frac{1}{2} \frac{m_b}{L_2} \left( \frac{4}{L_2^3} \right) \dot{y}_e^2 \int_0^{L_2/2} z^4 \left(3L_2 - 4z\right)^2 \, dz \]
\[ = \frac{1}{2} \frac{m_b}{35} \frac{13}{35} \dot{y}_e^2 \]

Equivalent mass at \( C = \frac{13}{35} m_b = 0.3714 m_b \)

e) 
\[ T = \frac{1}{2} I_{eq} \dot{\theta}^2 \]
\[ \Rightarrow \quad I_{eq} = \frac{13}{35} m_b (a+b)^2 + \left[ I_{G1} + m_1 \left( \frac{a+b}{2} \right)^2 \right] + I_{G1} \frac{a^2}{r_1^2} + mz a^2 \]